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Phil Dyke

An Introduction to Laplace Transforms and Fourier Series

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To Ottilie

Preface

This book has been primarily written for the student of mathematics who is in the second year or the early part of the third year of an undergraduate course. It will also be very useful for students of engineering and physical sciences for whom Laplace transforms continue to be an extremely useful tool. The book demands no more than an elementary knowledge of calculus and linear algebra of the type found in many first year mathematics modules for applied subjects. For mathematics majors and specialists, it is not the mathematics that will be challenging but the applications to the real world. The author is in the privileged position of having spent ten or so years outside mathematics in an engineering environment where the Laplace transform is used in anger to solve real problems, as well as spending rather more years within mathematics where accuracy and logic are of primary importance. This book is written unashamedly from the point of view of the applied mathematician.

The Laplace transform has a rather strange place in mathematics. There is no doubt that it is a topic worthy of study by applied mathematicians who have one eye on the wealth of applications; indeed it is often called Operational Calculus. However, because it can be thought of as specialist, it is often absent from the core of mathematics degrees, turning up as a topic in the second half of the second year when it comes in handy as a tool for solving certain breeds of differential equation. On the other hand, students of engineering (particularly the electrical and control variety) often meet Laplace transforms early in the first year and use them to solve engineering problems. It is for this kind of application that software packages (MATLAB©, for example) have been developed. These students are not expected to understand the theoretical basis of Laplace transforms. What I have attempted here is a mathematical look at the Laplace transform that demands no more of the reader than a knowledge of elementary calculus. The Laplace transform is seen in its typical guise as a handy tool for solving practical mathematical problems but, in addition, it is also seen as a particularly good vehicle for exhibiting fundamental ideas such as a mapping, linearity, an operator, a kernel and an image. These basic principals are covered in the first three chapters of the book. Alongside the Laplace transform, we develop the notion of Fourier series from first principals. Again no more than a working knowledge of trigonometry and elementary calculus is required from the student. Fourier series can be introduced via linear spaces, and exhibit properties such as orthogonality, linear independence and completeness which are so central to much of mathematics. This pure mathematics would be out of place in a text such as this, but Appendix C contains much of the background for those interested. In Chapter 4, Fourier series are introduced with an eye on the practical applications. Nevertheless it is still useful for the student to have encountered the notion of a vector space before tackling this chapter. Chapter 5 uses both Laplace transforms and Fourier series to solve partial differential equations. In Chapter 6, Fourier Transforms are discussed in their own right, and the link between these, Laplace transforms and Fourier series, is established. Finally, complex variable methods are introduced and used in the last chapter. Enough basic complex variable theory to understand the inversion of Laplace transforms is given here, but in order for Chapter 7 to be fully appreciated, the student will already need to have a working knowledge of complex variable theory before embarking on it. There are plenty of sophisticated software packages around these days, many of which will carry out Laplace transform integrals, the inverse, Fourier series and Fourier transforms. In solving real-life problems, the student will of course use one or more of these. However, this text introduces the basics; as necessary as a knowledge of arithmetic is to the proper use of a calculator.

At every age there are complaints from teachers that students in some respects fall short of the calibre once attained. In this present era, those who teach mathematics in higher education complain long and hard about the lack of stamina amongst today's students. If a problem does not come out in a few lines, the majority give up. I suppose the main cause of this is the computer/video age in which we live, in which amazing eye-catching images are available at the touch of a button. However, another contributory factor must be the decrease in the time devoted to algebraic manipulation, manipulating fractions etc. in mathematics in the 11–16 age range. Fortunately, the impact of this on the teaching of Laplace transforms and Fourier series is perhaps less than its impact in other areas of mathematics. (One thinks of mechanics and differential equations as areas where it will be greater.) Having said all this, the student is certainly encouraged to make use of good computer algebra packages (e.g. MAPLE©, MATHEMATICA©, DERIVE©, MACSYMA©) where appropriate. Of course, it is dangerous to rely totally on such software in much the same way as the existence of a good spell checker is no excuse for giving up the knowledge of being able to spell, but a good computer algebra package can facilitate factorisation, evaluation of expressions, performing long winded but otherwise routine calculus and algebra. The proviso is always that students must *understand* what they are doing before using packages as even modern day computers can still be extraordinarily dumb!

In writing this book, the author has made use of many previous works on the subject as well as unpublished lecture notes and examples. It is very difficult to know the precise source of examples especially when one has taught the material to students for some years, but the major sources can be found in the bibliography. I thank an anonymous referee for making many helpful suggestions. It is also a great pleasure to thank my daughter Ottilie whose familiarity and expertise with certain software was much appreciated and it is she who has produced many of the diagrams. The text itself has been produced using LATEX.

January 1999

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Preface to the Second Edition

Twelve years have elapsed since the first edition of this book, but a subject like Laplace transforms does not date. All of the book remains as relevant as it was at the turn of the millennium. I have taken the opportunity to correct annoying typing errors and other misprints. I would like to take this opportunity to thank everyone who has told me of the mistakes, especially those in the 1999 edition many of which owed a lot to the distraction of my duties as Head of School as well as my inexperience with LATEX. Here are the changes made; I have added a section on generalising Fourier series to the end of Chap. 4 and made slight alterations to Chap. 6 due to the presence of a new Chap. 7 on Wavelets and Signal Processing. The changes have developed both out of using the book as material for a second-year module in Mathematical Methods to year two undergraduate mathematicians for the past 6 years, and the increasing importance of digital signal processing. The end of the chapter exercises particularly those in the early chapters have undergone the equivalent of a good road test and have been improved accordingly. I have also lengthened Appendix B, the table of Laplace transforms, which looked thin in the first edition.

The biggest change from the first edition is of course the inclusion of the extra chapter. Although wavelets date from the early 1980s, their use only blossomed in the 1990s and did not form part of the typical undergraduate curriculum at the time of the first edition. Indeed the texts on wavelets I have quoted here in the bibliography are securely at graduate level, there are no others. What I have done is to introduce the idea of a wavelet (which is a pulse in time, zero outside a short range) and use Fourier methods to analyse it. The concepts involved sit nicely in a book at this level if treated as an application of Fourier series and transforms. I have not gone on to cover discrete transforms as this would move too far into signal processing and require statistical concepts that would be out of place to include here. The new chapter has been placed between Fourier Transforms (Chap. 6) and Complex Variables and Laplace Transforms (now Chap. 8).

In revising the rest of the book, I have made small additions but no subtractions, so the total length has increased a little.

Finally a word about software. I have resisted the inclusion of pseudocode or specific insets in MATLAB or MAPLE, even though the temptation was strong in relation to the new material on wavelets which owes its popularity largely to its widespread use in signal processing software. It remains my view that not only do these date quickly, but at this level the underlying principles covered here are best done without such embellishments. I use MAPLE and it is updated every year; it is now easy to use it in a cut and paste way, without code, to apply to Fourier series problems. It is a little more difficult (but not prohibitively so) to use cut and paste methods for Laplace and Fourier transforms calculations. Most students use software tools without fuss these days; so to overdo the specific references to software in a mathematics text now is a bit like making too many specific references to pencil and paper 50 years ago.

October 2013 Phil Dyke

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1.1 Introduction

As a discipline, mathematics encompasses a vast range of subjects. In pure mathematics an important concept is the idea of an axiomatic system whereby axioms are proposed and theorems are proved by invoking these axioms logically. These activities are often of little interest to the applied mathematician to whom the pure mathematics of algebraic structures will seem like tinkering with axioms for hours in order to prove the obvious. To the engineer, this kind of pure mathematics is even more of an anathema. The value of knowing about such structures lies in the ability to generalise the "obvious" to other areas. These generalisations are notoriously unpredictable and are often very surprising. Indeed, many say that there is no such thing as non-applicable mathematics, just mathematics whose application has yet to be found.

The Laplace transform expresses the conflict between pure and applied mathematics splendidly. There is a temptation to begin a book such as this on linear algebra outlining the theorems and properties of normed spaces. This would indeed provide a sound basis for future results. However most applied mathematicians and all engineers would probably turn off. On the other hand, engineering texts present the Laplace transform as a toolkit of results with little attention being paid to the underlying mathematical structure, regions of validity or restrictions. What has been decided here is to give a brief introduction to the underlying pure mathematical structures, enough it is hoped for the pure mathematician to appreciate what kind of creature the Laplace transform is, whilst emphasising applications and giving plenty of examples. The point of view from which this book is written is therefore definitely that of the applied mathematician. However, pure mathematical asides, some of which can be quite extensive, will occur. It remains the view of this author that Laplace transforms only come alive when they are used to solve real problems. Those who strongly disagree with this will find pure mathematics textbooks on integral transforms much more to their liking.

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The main area of pure mathematics needed to understand the fundamental properties of Laplace transforms is analysis and, to a lesser extent the normed vector space. Analysis, in particular integration, is needed from the start as it governs the existence conditions for the Laplace transform itself; however as is soon apparent, calculations involving Laplace transforms can take place without explicit knowledge of analysis. Normed vector spaces and associated linear algebra put the Laplace transform on a firm theoretical footing, but can be left until a little later in a book aimed at second year undergraduate mathematics students.

1.2 The Laplace Transform

The definition of the Laplace transform could hardly be more straightforward. Given a suitable function $F(t)$ the Laplace transform, written $f(s)$ is defined by

$$f(s) = \int_0^\infty F(t)e^{-st}dt.$$

This bald statement may satisfy most engineers, but not mathematicians. The question of what constitutes a "suitable function" will now be addressed. The integral on the right has infinite range and hence is what is called an improper integral. This too needs careful handling. The notation $\mathcal{L}\{F(t)\}$ is used to denote the Laplace transform of the function $F(t)$.

Another way of looking at the Laplace transform is as a mapping from points in the t domain to points in the s domain. Pictorially, Fig. 1.1 indicates this mapping process.

The time domain t will contain all those functions $F(t)$ whose Laplace transform exists, whereas the frequency domain s contains all the images $\mathcal{L}\{F(t)\}$. Another aspect of Laplace transforms that needs mentioning at this stage is that the variable s often has to take complex values. This means that $f(s)$ is a function of a complex variable, which in turn places restrictions on the (real) function $F(t)$ given that the improper integral must converge. Much of the analysis involved in dealing with the image of the function $F(t)$ in the s plane is therefore complex analysis which may be quite new to some readers.

As has been said earlier, engineers are quite happy to use Laplace transforms to help solve a variety of problems without questioning the convergence of the improper integrals. This goes for some applied mathematicians too. The argument seems to be on the lines that if it gives what looks a reasonable answer, then fine. In our view, this takes the engineer's maxim "if it ain't broke, don't fix it" too far. This is primarily a mathematics textbook, therefore in this opening chapter we shall be more mathematically explicit than is customary in books on Laplace transforms. In Chap. 4 there is some more pure mathematics when Fourier series are introduced. That is there for similar reasons. One mathematical question that ought to be asked concerns uniqueness. Given a function $F(t)$, its Laplace transform is surely unique

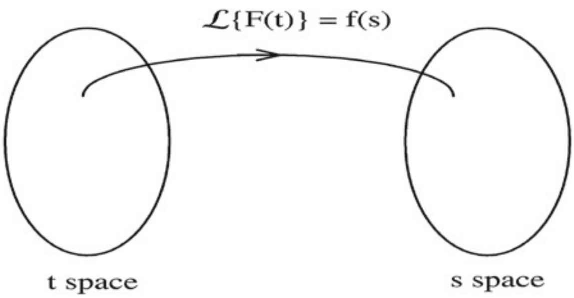


Fig. 1.1 The Laplace Transform as a mapping

from the well defined nature of the improper integral. However, is it possible for two different functions to have the same Laplace transform? To put the question a different but equivalent way, is there a function $N(t)$, not identically zero, whose Laplace transform is zero? For this function, called a *null* function, could be added to any suitable function and the Laplace transform would remain unchanged. Null functions do exist, but as long as we restrict ourselves to piecewise continuous functions this ceases to be a problem. Here is the definition of piecewise continuous:

Definition 1.1 *If an interval* $[0, t_0]$ *say can be partitioned into a finite number of subintervals* $[0, t_1], [t_1, t_2], [t_2, t_3], \ldots, [t_n, t_0]$ *with* $0, t_1, t_2, \ldots, t_n, t_0$ *an increasing sequence of times and such that a given function* $f(t)$ *is continuous in each of these subintervals but not necessarily at the end points themselves, then* $f(t)$ *is piecewise continuous in the interval* $[0, t_0]$.

Only functions that differ at a finite number of points have the same Laplace transform. If $F_1(t) = F(t)$ except at a finite number of points where they differ by finite values then $\mathcal{L}\{F_1(t)\} = \mathcal{L}\{F(t)\}$. We mention this again in the next chapter when the inverse Laplace transform is defined.

In this section, we shall examine the conditions for the existence of the Laplace transform in more detail than is usual. In engineering texts, the simple definition followed by an explanation of exponential order is all that is required. Those that are satisfied with this can virtually skip the next few paragraphs and go on study the elementary properties, Sect. 1.3. However, some may need to know enough background in terms of the integrals, and so we devote a little space to some fundamentals. We will need to introduce improper integrals, but let us first define the Riemann integral. It is the integral we know and love, and is defined in terms of limits of sums. The strict definition runs as follows:-

Let $F(x)$ be a function which is defined and is bounded in the interval $a \le x \le b$ and suppose that m and M are respectively the lower and upper bounds of $F(x)$ in this interval (written $[a, b]$ see Appendix C). Take a set of points

$$x_0 = a, x_1, x_2, \ldots, x_{r-1}, x_r, \ldots, x_n = b$$

and write $\delta_r = x_r - x_{r-1}$. Let M_r, m_r be the bounds of $F(x)$ in the subinterval (x_{r-1}, x_r) and form the sums

$$S = \sum_{r=1}^{n} M_r \delta_r$$

$$s = \sum_{r=1}^{n} m_r \delta_r.$$

These are called respectively the upper and lower Riemann sums corresponding to the mode of subdivision. It is certainly clear that $S \geq s$. There are a variety of ways that can be used to partition the interval (a, b) and each way will have (in general) different M_r and m_r leading to different S and s. Let M be the minimum of all possible M_r and m be the maximum of all possible m_r A lower bound or supremum for the set S is therefore $M(b-a)$ and an upper bound or infimum for the set s is $m(b-a)$. These bounds are of course rough. There are *exact* bounds for S and s, call them J and I respectively. If $I = J$, $F(x)$ is said to be Riemann integrable in (a, b) and the value of the integral is I or J and is denoted by

$$I = J = \int_a^b F(x)dx.$$

For the purist it turns out that the Riemann integral is not quite general enough, and the Stieltjes integral is actually required. However, we will not use this concept which belongs securely in specialist final stage or graduate texts.

The improper integral is defined in the obvious way by taking the limit:

$$\lim_{R\to\infty} \int_a^R F(x)dx = \int_a^\infty F(x)dx$$

provided $F(x)$ is continuous in the interval $a \leq x \leq R$ for every R, and the limit on the left exists. The parameter x is defined to take the increasing values from a to ∞. The lower limit a is normally 0 in the context of Laplace transforms. The condition $|F(x)| \leq Me^{\alpha x}$ is termed "$F(x)$ is of exponential order" and is, speaking loosely, quite a weak condition. All polynomial functions and (of course) exponential functions of the type e^{kx} (k constant) are included as well as bounded functions. Excluded functions are those that have singularities such as $\ln(x)$ or $1/(x-1)$ and functions that have a growth rate more rapid than exponential, for example e^{x^2}. Functions that have a finite number of finite discontinuities are also included. These have a special role in the theory of Laplace transforms so we will not dwell on them here: suffice to say that a function such as

$$F(x) = \begin{cases} 1 & 2n < x < 2n+1 \\ 0 & 2n+1 < x < 2n+2 \end{cases} \quad \text{where } n = 0, 1, \ldots$$

is one example. However, the function

$$F(x) = \begin{cases} 1 & x \text{ rational} \\ 0 & x \text{ irrational} \end{cases}$$

is excluded because although all the discontinuities are finite, there are infinitely many of them.

We shall now follow standard practice and use t (time) instead of x as the dummy variable.

1.3 Elementary Properties

The Laplace transform has many interesting and useful properties, the most fundamental of which is linearity. It is linearity that enables us to add results together to deduce other more complicated ones and is so basic that we state it as a theorem and prove it first.

Theorem 1.1 **(Linearity)** *If $F_1(t)$ and $F_2(t)$ are two functions whose Laplace transform exists, then*

$$\mathcal{L}\{aF_1(t) + bF_2(t)\} = a\mathcal{L}\{F_1(t)\} + b\mathcal{L}\{F_2(t)\}$$

where a and b are arbitrary constants.

Proof

$$\begin{aligned} \mathcal{L}\{aF_1(t) + bF_2(t)\} &= \int_0^\infty (aF_1 + bF_2)e^{-st}\,dt \\ &= \int_0^\infty \left(aF_1e^{-st} + bF_2e^{-st}\right)dt \\ &= * - a\int_0^\infty F_1e^{-st}dt + b\int_0^\infty F_2e^{-st}dt \\ &= a\mathcal{L}\{F_1(t)\} + b\mathcal{L}\{F_2(t)\} \end{aligned}$$

where we have assumed that

$$|F_1| \leq M_1e^{\alpha_1 t} \text{ and } |F_2| \leq M_2e^{\alpha_2 t}$$

so that

$$\begin{aligned} |aF_1 + bF_2| &\leq |a||F_1| + |b||F_2| \\ &\leq (|a|M_1 + |b|M_2)e^{\alpha_3 t} \end{aligned}$$

where $\alpha_3 = max\{\alpha_1, \alpha_2\}$. This proves the theorem.

□

Here we shall concentrate on those properties of the Laplace transform that do not involve the calculus. The first of these takes the form of another theorem because of its generality.

Theorem 1.2 (First Shift Theorem) *If it is possible to choose constants M and α such that $|F(t)| \leq Me^{\alpha t}$, that is $F(t)$ is of exponential order, then*

$$\mathcal{L}\{e^{-bt}F(t)\} = f(s+b)$$

provided $b \leq \alpha$. (In practice if $F(t)$ is of exponential order then the constant α can be chosen such that this inequality holds.)

Proof The proof is straightforward and runs as follows:-

$$\begin{aligned}
\mathcal{L}\{e^{-bt}F(t)\} &= \lim_{T\to\infty}\int_0^T e^{-st}e^{-bt}F(t)dt \\
&= \int_0^\infty e^{-st}e^{-bt}F(t)dt \text{ (as the limit exists)} \\
&= \int_0^\infty e^{-(s+b)t}F(t)dt \\
&= f(s+b).
\end{aligned}$$

This establishes the theorem. □

We shall make considerable use of this once we have established a few elementary Laplace transforms. This we shall now proceed to do.

Example 1.1 *Find the Laplace transform of the function $F(t) = t$.*

Solution Using the definition of Laplace transform,

$$\mathcal{L}(t) = \lim_{T\to\infty}\int_0^T te^{-st}dt.$$

Now, we have that

$$\begin{aligned}
\int_0^T te^{-st}dt &= \left[-\frac{t}{s}e^{-st}\right]_0^T - \int_0^T -\frac{1}{s}e^{-st}dt \\
&= -\frac{T}{s}e^{-sT} + \left[-\frac{1}{s^2}e^{-st}\right]_0^T \\
&= -\frac{T}{s}e^{-sT} - \frac{1}{s^2}e^{-sT} + \frac{1}{s^2}
\end{aligned}$$

this last expression tends to $\dfrac{1}{s^2}$ as $T \to \infty$.

Hence we have the result

$$\mathcal{L}(t) = \frac{1}{s^2}.$$

We can use this result to generalise as follows:

Corollary

$$\mathcal{L}(t^n) = \frac{n!}{s^{n+1}}, \quad n \text{ a positive integer.}$$

Proof The proof is straightforward:

$$\begin{aligned}
\mathcal{L}(t^n) &= \int_0^\infty t^n e^{-st} dt \quad \text{this time taking the limit straight away} \\
&= \left[-\frac{t^n}{s} e^{-st} \right]_0^\infty + \int_0^\infty \frac{nt^{n-1}}{s} e^{-st} dt \\
&= \frac{n}{s} \mathcal{L}(t^{n-1}).
\end{aligned}$$

If we put $n = 2$ in this recurrence relation we obtain

$$\mathcal{L}(t^2) = \frac{2}{s} \mathcal{L}(t) = \frac{2}{s^3}.$$

If we assume

$$\mathcal{L}(t^n) = \frac{n!}{s^{n+1}}$$

then

$$\mathcal{L}(t^{n+1}) = \frac{n+1}{s} \frac{n!}{s^{n+1}} = \frac{(n+1)!}{s^{n+2}}.$$

This establishes that

$$\mathcal{L}(t^n) = \frac{n!}{s^{n+1}}$$

by induction.

□

Example 1.2 *Find the Laplace transform of* $\mathcal{L}\{te^{at}\}$ *and deduce the value of* $\mathcal{L}\{t^n e^{at}\}$*, where a is a real constant and n a positive integer.*

Solution Using the first shift theorem with $b = -a$ gives

$$\mathcal{L}\{F(t)e^{at}\} = f(s-a)$$

so with

$$F(t) = t \text{ and } f = \frac{1}{s^2}$$

we get

$$\mathcal{L}\{te^{at}\} = \frac{1}{(s-a)^2}.$$

Using $F(t) = t^n$ the formula

$$\mathcal{L}\{t^n e^{at}\} = \frac{n!}{(s-a)^{n+1}}$$

follows.

Later, we shall generalise this formula further, extending to the case where n is not an integer.

We move on to consider the Laplace transform of trigonometric functions. Specifically, we shall calculate $\mathcal{L}\{\sin t\}$ and $\mathcal{L}\{\cos t\}$. It is unfortunate, but the Laplace transform of the other common trigonometric functions tan, cot, csc and sec do not exist as they all have singularities for finite t. The condition that the function $F(t)$ has to be of exponential order is not obeyed by any of these singular trigonometric functions as can be seen, for example, by noting that

$$|e^{-at}\tan t| \to \infty \text{ as } t \to \pi/2$$

and

$$|e^{-at}\cot t| \to \infty \text{ as } t \to 0$$

for all values of the constant a. Similarly neither csc nor sec are of exponential order.

In order to find the Laplace transform of $\sin t$ and $\cos t$ it is best to determine $\mathcal{L}(e^{it})$ where $i = \sqrt{(-1)}$. The function e^{it} is complex valued, but it is both continuous and bounded for all t so its Laplace transform certainly exists. Taking the Laplace transform,

$$\begin{aligned}
\mathcal{L}(e^{it}) &= \int_0^\infty e^{-st}e^{it}dt \\
&= \int_0^\infty e^{t(i-s)}dt \\
&= \left[\frac{e^{(i-s)t}}{i-s}\right]_0^\infty \\
&= \frac{1}{s-i} \\
&= \frac{s}{s^2+1} + i\frac{1}{s^2+1}.
\end{aligned}$$

Now,

$$\begin{aligned}\mathcal{L}(e^{it}) &= \mathcal{L}(\cos t + i \sin t)\\ &= \mathcal{L}(\cos t) + i\mathcal{L}(\sin t).\end{aligned}$$

Equating real and imaginary parts gives the two results

$$\mathcal{L}(\cos t) = \frac{s}{s^2+1}$$

and

$$\mathcal{L}(\sin t) = \frac{1}{s^2+1}.$$

The linearity property has been used here, and will be used in future without further comment.

Given that the restriction on the type of function one can Laplace transform is weak, i.e. it has to be of exponential order and have at most a finite number of finite jumps, one can find the Laplace transform of any polynomial, any combination of polynomial with sinusoidal functions and combinations of these with exponentials (provided the exponential functions grow at a rate $\leq e^{at}$ where a is a constant). We can therefore approach the problem of calculating the Laplace transform of power series. It is possible to take the Laplace transform of a power series term by term as long as the series uniformly converges to a piecewise continuous function. We shall investigate this further later; meanwhile let us look at the Laplace transform of functions that are not even continuous.

Functions that are not continuous occur naturally in branches of electrical and control engineering, and in the software industry. One only has to think of switches to realise how widespread discontinuous functions are throughout electronics and computing.

Example 1.3 *Find the Laplace transform of the function represented by $F(t)$ where*

$$F(t) = \begin{cases} t & 0 \leq t < t_0 \\ 2t_0 - t & t_0 \leq t \leq 2t_0 \\ 0 & t > 2t_0. \end{cases}$$

Solution This function is of the "saw-tooth" variety that is quite common in electrical engineering. There is no question that it is of exponential order and that

$$\int_0^\infty e^{-st} F(t)dt$$

exists and is well defined. $F(t)$ is continuous but not differentiable. This is not troublesome. Carrying out the calculation is a little messy and the details can be checked using MAPLE.

$$
\begin{aligned}
\mathcal{L}(F(t)) &= \int_0^\infty e^{-st}F(t)dt \\
&= \int_0^{t_0} te^{-st}dt + \int_{t_0}^{2t_0}(2t_0 - t)e^{-st}dt \\
&= \left[-\frac{t}{s}e^{-st}\right]_0^{t_0} + \int_0^{t_0}\frac{1}{s}e^{-st}dt + \left[-\frac{2t_0 - t}{s}e^{-st}\right]_{t_0}^{2t_0} - \int_{t_0}^{2t_0}\frac{1}{s}e^{-st}dt \\
&= -\frac{t_0}{s}e^{-st_0} - \frac{1}{s^2}\left[e^{-st}\right]_0^{t_0} + \frac{t_0}{s}e^{-st_0} + \frac{1}{s^2}\left[e^{-st}\right]_{t_0}^{2t_0} \\
&= \frac{1}{s^2}\left[e^{-st_0} - 1\right] + \frac{1}{s^2}\left[e^{-2st_0} - e^{-st_0}\right] \\
&= \frac{1}{s^2}\left[1 - 2e^{-st_0} + e^{-2st_0}\right] \\
&= \frac{1}{s^2}\left[1 - e^{-st_0}\right]^2 \\
&= \frac{4}{s^2}e^{-st_0}\sinh^2(\frac{1}{2}st_0).
\end{aligned}
$$

A bit later we shall investigate in more detail the properties of discontinuous functions such as the Heaviside unit step function. As an introduction to this, let us do the following example.

Example 1.4 *Determine the Laplace transform of the step function $F(t)$ defined by*

$$
F(t) = \begin{cases} 0 & 0 \le t < t_0 \\ a & t \ge t_0. \end{cases}
$$

Solution $F(t)$ itself is bounded, so there is no question that it is also of exponential order. The Laplace transform of $F(t)$ is therefore

$$
\begin{aligned}
\mathcal{L}(F(t)) &= \int_0^\infty e^{-st}F(t)dt \\
&= \int_{t_0}^\infty ae^{-st}dt \\
&= \left[-\frac{a}{s}e^{-st}\right]_{t_0}^\infty \\
&= \frac{a}{s}e^{-st_0}.
\end{aligned}
$$

Here is another useful general result; we state it as a theorem.

Theorem 1.3 *If $\mathcal{L}(F(t)) = f(s)$ then $\mathcal{L}(tF(t)) = -\dfrac{d}{ds}f(s)$ and in general $\mathcal{L}(t^nF(t)) = (-1)^n\dfrac{d^n}{ds^n}f(s)$.*

Proof Let us start with the definition of Laplace transform

$$\mathcal{L}(F(t)) = \int_0^\infty e^{-st} F(t)dt$$

and differentiate this with respect to s to give

$$\begin{aligned}\frac{df}{ds} &= \frac{d}{ds}\int_0^\infty e^{-st} F(t)dt \\ &= \int_0^\infty -te^{-st} F(t)dt\end{aligned}$$

assuming absolute convergence to justify interchanging differentiation and (improper) integration. Hence

$$\mathcal{L}(tF(t)) = -\frac{d}{ds} f(s).$$

One can now see how to progress by induction. Assume the result holds for n, so that

$$\mathcal{L}(t^n F(t)) = (-1)^n \frac{d^n}{ds^n} f(s)$$

and differentiate both sides with respect to s (assuming all appropriate convergence properties) to give

$$\int_0^\infty -t^{n+1} e^{-st} F(t)dt = (-1)^n \frac{d^{n+1}}{ds^{n+1}} f(s)$$

or

$$\int_0^\infty t^{n+1} e^{-st} F(t)dt = (-1)^{n+1} \frac{d^{n+1}}{ds^{n+1}} f(s).$$

So

$$\mathcal{L}(t^{n+1} F(t)) = (-1)^{n+1} \frac{d^{n+1}}{ds^{n+1}} f(s)$$

which establishes the result by induction. □

Example 1.5 *Determine the Laplace transform of the function $t \sin t$.*

Solution To evaluate this Laplace transform we use Theorem 1.3 with $f(t) = \sin t$. This gives

$$\mathcal{L}\{t \sin t\} = -\frac{d}{ds}\left\{\frac{1}{1+s^2}\right\} = \frac{2s}{(1+s^2)^2}$$

which is the required result.

1.4 Exercises

1. For each of the following functions, determine which has a Laplace transform. If it exists, find it; if it does not, say briefly why.

   (a) $\ln t$, (b) e^{3t}, (c) e^{t^2}, (d) $e^{1/t}$, (e) $1/t$,

   (f) $f(t) = \begin{cases} 1 \text{ if } t \text{ is even} \\ 0 \text{ if } t \text{ is odd.} \end{cases}$

2. Determine from first principles the Laplace transform of the following functions:-

   (a) e^{kt}, (b) t^2, (c) $\cosh(t)$.

3. Find the Laplace transforms of the following functions:-

   (a) t^2e^{-3t}, (b) $4t + 6e^{4t}$, (c) $e^{-4t}\sin(5t)$.

4. Find the Laplace transform of the function $F(t)$, where $F(t)$ is given by
   $F(t) = \begin{cases} t & 0 \le t < 1 \\ 2-t & 1 \le t < 2 \\ 0 & \text{otherwise.} \end{cases}$
5. Use the property of Theorem 1.3 to determine the following Laplace transforms

   (a) te^{2t}, (b) $t\cos(t)$, (c) $t^2\cos(t)$.

6. Find the Laplace transforms of the following functions:-

   (a) $\sin(\omega t + \phi)$, (b) $e^{5t}\cosh(6t)$.

7. If $G(at + b) = F(t)$ determine the Laplace transform of G in terms of $\mathcal{L}\{F\} = \bar{f}(s)$ and a finite integral.
8. Prove the following change of scale result:-

   $$\mathcal{L}\{F(at)\} = \frac{1}{a} f\left(\frac{s}{a}\right).$$

   Hence evaluate the Laplace transforms of the two functions

   (a) $t\cos(6t)$, (b) $t^2\cos(7t)$.

Chapter 2
Further Properties of the Laplace Transform

2.1 Real Functions

Sometimes, a function $F(t)$ represents a natural or engineering process that has no obvious starting value. Statisticians call this a *time series*. Although we shall not be considering $F(t)$ as stochastic, it is nevertheless worth introducing a way of "switching on" a function. Let us start by finding the Laplace transform of a step function the name of which pays homage to the pioneering electrical engineer Oliver Heaviside (1850–1925). The formal definition runs as follows.

Definition 2.1 *Heaviside's unit step function, or simply the unit step function, is defined as*

$$H(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0. \end{cases}$$

Since $H(t)$ is precisely the same as 1 for $t > 0$, the Laplace transform of $H(t)$ must be the same as the Laplace transform of 1, i.e. $1/s$. The switching on of an arbitrary function is achieved simply by multiplying it by the standard function $H(t)$, so if $F(t)$ is given by the function shown in Fig. 2.1 and we multiply this function by the Heaviside unit step function $H(t)$ to obtain $H(t)F(t)$, Fig. 2.2 results. Sometimes it is necessary to define what is called the *two sided* Laplace transform

$$\int_{-\infty}^{\infty} e^{-st} F(t)dt$$

which makes a great deal of mathematical sense. However the additional problems that arise by allowing negative values of t are severe and limit the use of the two sided Laplace transform. For this reason, the two sided transform will not be pursued here.

P. Dyke, *An Introduction to Laplace Transforms and Fourier Series*,
Springer Undergraduate Mathematics Series, DOI: 10.1007/978-1-4471-6395-4_2,

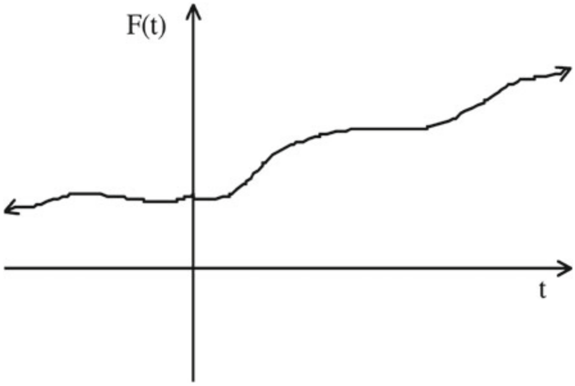


Fig. 2.1 $F(t)$, a function with no well defined starting value

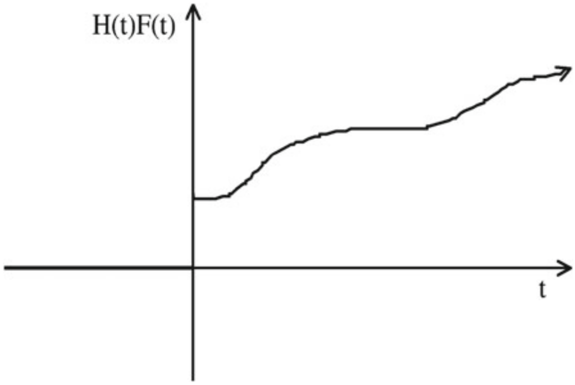


Fig. 2.2 $H(t)F(t)$, the function is now zero before $t = 0$

2.2 Derivative Property of the Laplace Transform

Suppose a differentiable function $F(t)$ has Laplace transform $f(s)$, we can find the Laplace transform

$$\mathcal{L}\{F'(t)\} = \int_0^\infty e^{-st} F'(t)dt$$

of its derivative $F'(t)$ through the following theorem.

Theorem 2.1

$$\mathcal{L}\{F'(t)\} = \int_0^\infty e^{-st} F'(t)dt = -F(0) + sf(s).$$

Proof Integrating by parts once gives

$$\begin{aligned}\mathcal{L}\{F'(t)\} &= \left[F(t)e^{-st}\right]_0^\infty + \int_0^\infty se^{-st}F(t)dt \\ &= -F(0) + sf(s)\end{aligned}$$

where $F(0)$ is the value of $F(t)$ at $t = 0$. □

This is an important result and lies behind future applications that involve solving linear differential equations. The key property is that the transform of a derivative $F'(t)$ does not itself involve a derivative, only $-F(0) + sf(s)$ which is an algebraic expression involving $f(s)$. The downside is that the value $F(0)$ is required. Effectively, an integration has taken place and the constant of integration is $F(0)$. Later, this is exploited further through solving differential equations. Later still in this text, partial differential equations are solved, and wavelets are introduced. Let us proceed here by finding the Laplace transform of the second derivative of $F(t)$. We also state this in the form of a theorem.

Theorem 2.2 *If $F(t)$ is a twice differentiable function of t then*

$$\mathcal{L}\{F''(t)\} = s^2 f(s) - sF(0) - F'(0).$$

Proof The proof is unremarkable and involves integrating by parts twice. Here are the details.

$$\begin{aligned}\mathcal{L}\{F''(t)\} &= \int_0^\infty e^{-st}F''(t)dt \\ &= \left[F'(t)e^{-st}\right]_0^\infty + \int_0^\infty se^{-st}F'(t)dt \\ &= -F'(0) + \left[sF(t)e^{-st}\right]_0^\infty + \int_0^\infty s^2e^{-st}F(t)dt \\ &= -F'(0) - sF(0) + s^2f(s) \\ &= s^2f(s) - sF(0) - F'(0).\end{aligned}$$

□

The general result, proved by induction, is

$$\mathcal{L}\{F^{(n)}(t)\} = s^n f(s) - s^{n-1}F(0) - s^{n-2}F'(0) - \cdots - F^{(n-1)}(0)$$

where n is a positive integer. Note the appearance of n constants on the right hand side. This of course is the result of integrating this number of times.

This result, as we have said, has wide application so it is worth getting to know. Consider the result

$$\mathcal{L}(\sin(\omega t)) = \frac{\omega}{s^2 + \omega^2}.$$

Now,

$$\frac{d}{dt}(\sin(\omega t)) = \omega \cos(\omega t)$$

so using the formula

$$\mathcal{L}(F'(t)) = sf(s) - F(0)$$

with $F(t) = \sin(\omega t)$ we have

$$\mathcal{L}\{\omega \cos(\omega t)\} = s\frac{\omega}{s^2 + \omega^2} - 0$$

so

$$\mathcal{L}\{\cos(\omega t)\} = \frac{s}{s^2 + \omega^2}$$

another standard result.

Another appropriate quantity to find at this point is the determination of the value of the Laplace transform of

$$\int_0^t F(u)du.$$

First of all, the function $F(t)$ must be integrable in such a way that

$$g(t) = \int_0^t F(u)du$$

is of exponential order. From this definition of $g(t)$ it is immediately apparent that $g(0) = 0$ and that $g'(t) = F(t)$. This latter result is called the fundamental theorem of the calculus. We can now use the result

$$\mathcal{L}\{g'(t)\} = s\bar{g}(s) - g(0)$$

to obtain

$$\mathcal{L}\{F(t)\} = f(s) = s\bar{g}(s)$$

where we have written $\mathcal{L}\{g(t)\} = \bar{g}(s)$. Hence

$$\bar{g}(s) = \frac{f(s)}{s}$$

which finally gives the result

$$\mathcal{L}\left(\int_0^t F(u)du\right) = \frac{f(s)}{s}.$$

The following result is also useful and can be stated in the form of a theorem.

Theorem 2.3 *If* $\mathcal{L}(F(t)) = f(s)$ *then* $\mathcal{L}\left\{\frac{F(t)}{t}\right\} = \int_s^\infty f(u)du$, *assuming that*

$$\mathcal{L}\left\{\frac{F(t)}{t}\right\} \to 0 \text{ as } s \to \infty.$$

Proof Let $G(t)$ be the function $F(t)/t$, so that $F(t) = tG(t)$. Using the property

$$\mathcal{L}\{tG(t)\} = -\frac{d}{ds}\mathcal{L}\{G(t)\}$$

we deduce that

$$f(s) = \mathcal{L}\{F(t)\} = -\frac{d}{ds}\mathcal{L}\left\{\frac{F(t)}{t}\right\}.$$

Integrating both sides of this with respect to s from s to ∞ gives

$$\int_s^\infty f(u)du = \left[-\mathcal{L}\left\{\frac{F(t)}{t}\right\}\right]_s^\infty = \mathcal{L}\left\{\frac{F(t)}{t}\right\}\Big|_s = \mathcal{L}\left\{\frac{F(t)}{t}\right\}$$

since

$$\mathcal{L}\left\{\frac{F(t)}{t}\right\} \to 0 \text{ as } s \to \infty$$

which completes the proof. □

The function

$$Si(t) = \int_0^t \frac{\sin u}{u}du$$

defines the Sine Integral function which occurs in the study of optics. The formula for its Laplace transform can now be easily derived as follows.

Let $F(t) = \sin t$ in the result

$$\mathcal{L}\left(\frac{F(t)}{t}\right) = \int_s^\infty f(u)du$$

to give

$$\begin{aligned}\mathcal{L}\left(\frac{\sin t}{t}\right) &= \int_s^\infty \frac{du}{1+u^2}\\ &= \left[\tan^{-1}(u)\right]_s^\infty\\ &= \frac{\pi}{2} - \tan^{-1}(s)\\ &= \tan^{-1}\left(\frac{1}{s}\right).\end{aligned}$$

We now use the result

$$\mathcal{L}\left(\int_0^t F(u)du\right) = \frac{f(s)}{s}$$

to deduce that

$$\mathcal{L}\left(\int_0^t \frac{\sin u}{u} du\right) = \mathcal{L}\{Si(t)\} = \frac{1}{s}\tan^{-1}\left(\frac{1}{s}\right).$$

2.3 Heaviside's Unit Step Function

As promised earlier, we devote this section to exploring some properties of Heaviside's unit step function $H(t)$. The Laplace transform of $H(t)$ has already been shown to be the same as the Laplace transform of 1, i.e. $1/s$. The Laplace transform of $H(t - t_0)$, $t_0 > 0$, is a little more enlightening:

$$\mathcal{L}\{H(t - t_0)\} = \int_0^\infty H(t - t_0)e^{-st}dt.$$

Now, since $H(t - t_0) = 0$ for $t < t_0$ this Laplace transform is

$$\mathcal{L}\{H(t - t_0)\} = \int_{t_0}^\infty e^{-st}dt = \left[-\frac{e^{-st}}{s}\right]_{t_0}^\infty = \frac{e^{-st_0}}{s}.$$

This result is generalised through the following theorem.

Theorem 2.4 **(Second Shift Theorem)** *If $F(t)$ is a function of exponential order in t then*

$$\mathcal{L}\{H(t - t_0)F(t - t_0)\} = e^{-st_0} f(s)$$

where $f(s)$ is the Laplace transform of $F(t)$.

Proof This result is proved by direct integration.

$$\begin{aligned}
\mathcal{L}\{H(t - t_0)F(t - t_0)\} &= \int_0^\infty H(t - t_0)F(t - t_0)e^{-st}dt \\
&= \int_{t_0}^\infty F(t - t_0)e^{-st}dt \text{ (by definition of } H) \\
&= \int_0^\infty F(u)e^{-s(u+t_0)}du \text{ (writing } u = t - t_0) \\
&= e^{-st_0} f(s).
\end{aligned}$$

This establishes the theorem. □

The only condition on $F(t)$ is that it is a function that is of exponential order which means of course that it is free from singularities for $t > t_0$. The principal use of this theorem is that it enables us to determine the Laplace transform of a function that is switched on at time $t = t_0$. Here is a straightforward example.

Example 2.1 *Determine the Laplace transform of the sine function switched on at time $t = 3$.*

Solution The sine function required that starts at $t = 3$ is $S(t)$ where

$$S(t) = \begin{cases} \sin t & t \geq 3 \\ 0 & t < 3. \end{cases}$$

We can use the Heaviside step function to write

$$S(t) = H(t-3)\sin t.$$

The second shift theorem can then be used by utilising the summation formula

$$\sin t = \sin(t-3+3) = \sin(t-3)\cos(3) + \cos(t-3)\sin(3)$$

so

$$\mathcal{L}\{S(t)\} = \mathcal{L}\{H(t-3)\sin(t-3)\}\cos(3) + \mathcal{L}\{H(t-3)\cos(t-3)\}\sin(3).$$

This may seem a strange step to take, but in order to use the second shift theorem it is essential to get the arguments of both the Heaviside function and the target function in the question the same; in this case $(t-3)$. We can now use the second shift theorem directly to give

$$\mathcal{L}\{S(t)\} = e^{-3s}\cos(3)\frac{1}{s^2+1} + e^{-3s}\sin(3)\frac{s}{s^2+1}$$

or

$$\mathcal{L}\{S(t)\} = (\cos 3 + s\sin 3)e^{-3s}/(s^2+1).$$

2.4 Inverse Laplace Transform

Virtually all operations have inverses. Addition has subtraction, multiplication has division, differentiation has integration. The Laplace transform is no exception, and we can define the *Inverse* Laplace transform as follows.

Definition 2.2 *If $F(t)$ has the Laplace transform $f(s)$, that is*

$$\mathcal{L}\{F(t)\} = f(s)$$

then the inverse Laplace transform is defined by

$$\mathcal{L}^{-1}\{f(s)\} = F(t)$$

and is unique apart from null functions.

Perhaps the most important property of the inverse transform to establish is its linearity. We state this as a theorem.

Theorem 2.5 *The inverse Laplace transform is linear, i.e.*

$$\mathcal{L}^{-1}\{af_1(s) + bf_2(s)\} = a\mathcal{L}^{-1}\{f_1(s)\} + b\mathcal{L}^{-1}\{f_2(s)\}.$$

Proof Linearity is easily established as follows. Since the Laplace transform is linear, we have for suitably well behaved functions $F_1(t)$ and $F_2(t)$:

$$\mathcal{L}\{aF_1(t) + bF_2(t)\} = a\mathcal{L}\{F_1(t)\} + b\mathcal{L}\{F_2(t)\} = af_1(s) + bf_2(s).$$

Taking the inverse Laplace transform of this expression gives

$$aF_1(t) + bF_2(t) = \mathcal{L}^{-1}\{af_1(s) + bf_2(s)\}$$

which is the same as

$$a\mathcal{L}^{-1}\{f_1(s)\} + b\mathcal{L}^{-1}\{f_2(s)\} = \mathcal{L}^{-1}\{af_1(s) + bf_2(s)\}$$

and this has established linearity of $\mathcal{L}^{-1}\{f(s)\}$. □

Another important property is uniqueness. It has been mentioned that the Laplace transform was indeed unique apart from null functions (functions whose Laplace transform is zero). It follows immediately that the inverse Laplace transform is also unique apart from the possible addition of null functions. These take the form of isolated values and can be discounted for all practical purposes.

As is quite common with inverse operations there is no systematic method of determining inverse Laplace transforms. The calculus provides a good example where there are plenty of systematic rules for differentiation: the product rule, the quotient rule, the chain rule. However by contrast there are no systematic rules for the inverse operation, integration. If we have an integral to find, we may try substitution or integration by parts, but there is no guarantee of success. Indeed, the integral may not be possible to express in terms of elementary functions. Derivatives that exist can always be found by using the rules; this is not so for integrals. The situation regarding the Laplace transform is not quite the same in that it may not be possible to find $\mathcal{L}\{F(t)\}$ explicitly because it is an integral. There is certainly no guarantee of being able to find $\mathcal{L}^{-1}\{f(s)\}$ and we have to devise various methods of trying so to do. For example, given an arbitrary function of s there is no guarantee whatsoever that a function of t can be found that is its inverse Laplace transform. One necessary condition for example is that the function of s must tend to zero as $s \to \infty$. When we are certain that a function of s has arisen from a Laplace transform, there are techniques and theorems that can help us invert it. Partial fractions simplify rational functions and can help identify standard forms (the exponential and trigonometric functions for example), then there are the shift theorems which we have just met which extend further the repertoire of standard forms. Engineering texts spend a considerable amount of space building up a library of specific inverse Laplace transforms and to ways of extending these via the calculus. To a certain extent we need to do this too. Therefore we next do some reasonably elementary examples. Note that in Appendix B there is a list of some inverse Laplace transforms.

Example 2.2 *Use partial fractions to determine*

$$\mathcal{L}^{-1}\left\{\frac{a}{s^2-a^2}\right\}$$

Solution Noting that

$$\frac{a}{s^2-a^2} = \frac{1}{2}\left[\frac{1}{s-a} - \frac{1}{s+a}\right]$$

gives straight away that

$$\mathcal{L}^{-1}\left\{\frac{a}{s^2-a^2}\right\} = \frac{1}{2}(e^{at} - e^{-at}) = \sinh(at).$$

The first shift theorem has been used on each of the functions $1/(s-a)$ and $1/(s+a)$ together with the standard result $\mathcal{L}^{-1}\{1/s\} = 1$. Here is another example.

Example 2.3 *Determine the value of*

$$\mathcal{L}^{-1}\left\{\frac{s^2}{(s+3)^3}\right\}.$$

Solution Noting the standard partial fraction decomposition

$$\frac{s^2}{(s+3)^3} = \frac{1}{s+3} - \frac{6}{(s+3)^2} + \frac{9}{(s+3)^3}$$

we use the first shift theorem on each of the three terms in turn to give

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{s^2}{(s+3)^3}\right\} &= \mathcal{L}^{-1}\frac{1}{s+3} - \mathcal{L}^{-1}\frac{6}{(s+3)^2} + \mathcal{L}^{-1}\frac{9}{(s+3)^3} \\ &= e^{-3t} - 6te^{-3t} + \frac{9}{2}t^2e^{-3t}\end{aligned}$$

where we have used the linearity property of the $\mathcal{L}^{-1}$ operator. Finally, we do the following four-in-one example to hone our skills.

Example 2.4 *Determine the following inverse Laplace transforms*

$$(a)\ \mathcal{L}^{-1}\frac{(s+3)}{s(s-1)(s+2)};\ (b)\ \mathcal{L}^{-1}\frac{(s-1)}{s^2+2s-8};\ (c)\ \mathcal{L}^{-1}\frac{3s+7}{s^2-2s+5};\ (d)\ \mathcal{L}^{-1}\frac{e^{-7s}}{(s+3)^3}.$$

Solution All of these problems are tackled in a similar way, by decomposing the expression into partial fractions, using shift theorems, then identifying the simplified expressions with various standard forms.

(a) Using partial fraction decomposition and not dwelling on the detail we get

$$\frac{s+3}{s(s-1)(s+2)} = -\frac{3}{2s} + \frac{4}{3(s-1)} + \frac{1}{6(s+2)}.$$

Hence, operating on both sides with the inverse Laplace transform operator gives

$$\begin{aligned}\mathcal{L}^{-1}\frac{s+3}{s(s-1)(s+2)} &= -\mathcal{L}^{-1}\frac{3}{2s} + \mathcal{L}^{-1}\frac{4}{3(s-1)} + \mathcal{L}^{-1}\frac{1}{6(s+2)}\\ &= -\frac{3}{2}\mathcal{L}^{-1}\frac{1}{s} + \frac{4}{3}\mathcal{L}^{-1}\frac{1}{s-1} + \frac{1}{6}\mathcal{L}^{-1}\frac{1}{s+2}\end{aligned}$$

using the linearity property of $\mathcal{L}^{-1}$ once more. Finally, using the standard forms, we get

$$\mathcal{L}^{-1}\left\{\frac{s+3}{s(s-1)(s+2)}\right\} = -\frac{3}{2} + \frac{4}{3}e^{t} + \frac{1}{6}e^{-2t}.$$

(b) The expression

$$\frac{s-1}{s^2+2s-8}$$

is factorised to

$$\frac{s-1}{(s+4)(s-2)}$$

which, using partial fractions is

$$\frac{1}{6(s-2)} + \frac{5}{6(s+4)}.$$

Therefore, taking inverse Laplace transforms gives

$$\mathcal{L}^{-1}\frac{s-1}{s^2+2s-8} = \frac{1}{6}e^{2t} + \frac{5}{6}e^{-4t}.$$

(c) The denominator of the rational function

$$\frac{3s+7}{s^2-2s+5}$$

does not factorise. In this case we use completing the square and standard trigonometric forms as follows:

$$\frac{3s+7}{s^2-2s+5} = \frac{3s+7}{(s-1)^2+4} = \frac{3(s-1)+10}{(s-1)^2+4}.$$

So

$$\begin{aligned}\mathcal{L}^{-1}\frac{3s+7}{s^2-2s+5} &= 3\mathcal{L}^{-1}\frac{(s-1)}{(s-1)^2+4} + 5\mathcal{L}^{-1}\frac{2}{(s-1)^2+4}\\ &= 3e^t\cos(2t) + 5e^t\sin(2t).\end{aligned}$$

Again, the first shift theorem has been used.

(d) The final inverse Laplace transform is slightly different. The expression

$$\frac{e^{-7s}}{(s-3)^3}$$

contains an exponential in the numerator, therefore it is expected that the second shift theorem will have to be used. There is a little "fiddling" that needs to take place here. First of all, note that

$$\mathcal{L}^{-1}\frac{1}{(s-3)^3} = \frac{1}{2}t^2e^{3t}$$

using the first shift theorem. So

$$\mathcal{L}^{-1}\frac{e^{-7s}}{(s-3)^3} = \begin{cases}\frac{1}{2}(t-7)^2e^{3(t-7)} & t>7\\ 0 & 0\le t\le 7.\end{cases}$$

Of course, this can succinctly be expressed using the Heaviside unit step function as

$$\frac{1}{2}H(t-7)(t-7)^2e^{3(t-7)}.$$

We shall get more practice at this kind of inversion exercise, but you should try your hand at a few of the exercises at the end.

2.5 Limiting Theorems

In many branches of mathematics there is a necessity to solve differential equations. Later chapters give details of how some of these equations can be solved by using Laplace transform techniques. Unfortunately, it is sometimes the case that it is not possible to invert $f(s)$ to retrieve the desired solution to the original problem. Numerical inversion techniques are possible and these can be found in some software packages, especially those used by control engineers. Insight into the behaviour of the solution can be deduced without actually solving the differential equation by examining the asymptotic character of $f(s)$ for small s or large s. In fact, it is often very useful to determine this asymptotic behaviour without solving the equation, even when exact solutions are available as these solutions are often complex and difficult to obtain let alone interpret. In this section two theorems that help us to find this asymptotic behaviour are investigated.

Theorem 2.6 (Initial Value) *If the indicated limits exist then*

$$\lim_{t\to 0} F(t) = \lim_{s\to\infty} sf(s).$$

(The left hand side is $F(0)$ *of course, or* $F(0+)$ *if* $\lim_{t\to 0} F(t)$ *is not unique.)*

Proof We have already established that

$$\mathcal{L}\{F'(t)\} = sf(s) - F(0). \tag{2.1}$$

However, if $F'(t)$ obeys the usual criteria for the existence of the Laplace transform, that is $F'(t)$ is of exponential order and is piecewise continuous, then

$$\begin{aligned}\left|\int_0^\infty e^{-st}F'(t)dt\right| &\le \int_0^\infty |e^{-st}F'(t)|dt\\ &\le \int_0^\infty e^{-st}e^{Mt}dt\\ &= -\frac{1}{M-s} \to 0 \text{ as } s\to\infty.\end{aligned}$$

Thus letting $s\to\infty$ in Eq. (2.1) yields the result. □

Theorem 2.7 (Final Value) *If the limits indicated exist, then*

$$\lim_{t\to\infty} F(t) = \lim_{s\to 0} sf(s).$$

Proof Again we start with the formula for the Laplace transform of the derivative of $F(t)$

$$\mathcal{L}\{F'(t)\} = \int_0^\infty e^{-st} F'(t)dt = sf(s) - F(0) \tag{2.2}$$

this time writing the integral out explicitly. The limit of the integral as $s \to 0$ is

$$\begin{aligned}
\lim_{s\to 0} \int_0^\infty e^{-st} F'(t)dt &= \lim_{s\to 0} \lim_{T\to\infty} \int_0^T e^{-st} F'(t)dt \\
&= \lim_{s\to 0} \lim_{T\to\infty} \{e^{-sT} F(T) - F(0)\} \\
&= \lim_{T\to\infty} F(T) - F(0) \\
&= \lim_{t\to\infty} F(t) - F(0).
\end{aligned}$$

Thus we have, using Eq. (2.2),

$$\lim_{t\to\infty} F(t) - F(0) = \lim_{s\to 0} sf(s) - F(0)$$

from which, on cancellation of $-F(0)$, the theorem follows. □

Since the improper integral converges independently of the value of s and all limits exist (a priori assumption), it is therefore correct to have assumed that the order of the two processes (taking the limit and performing the integral) can be exchanged. (This has in fact been demonstrated explicitly in this proof.)

Suppose that the function $F(t)$ can be expressed as a power series as follows

$$F(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n + \cdots .$$

If we assume that the Laplace transform of $F(t)$ exists, $F(t)$ is of exponential order and is piecewise continuous. If, further, we assume that the power series for $F(t)$ is absolutely and uniformly convergent the Laplace transform can be applied term by term

$$\begin{aligned}
\mathcal{L}\{F(t)\} = f(s) &= \mathcal{L}\{a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n + \cdots\} \\
&= a_0 \mathcal{L}\{1\} + a_1 \mathcal{L}\{t\} + a_2 \mathcal{L}\{t^2\} + \cdots + a_n \mathcal{L}\{t^n\} + \cdots
\end{aligned}$$

provided the transformed series is convergent. Using the standard form

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$$

the right hand side becomes

$$\frac{a_0}{s} + \frac{a_1}{s^2} + \frac{2a_2}{s^3} + \cdots + \frac{n! a_n}{s^{n+1}} + \cdots .$$

Hence

$$f(s) = \frac{a_0}{s} + \frac{a_1}{s^2} + \frac{2a_2}{s^3} + \cdots + \frac{n!a_n}{s^{n+1}} + \cdots .$$

Example 2.5 *Demonstrate the initial and final value theorems using the function $F(t) = e^{-t}$. Expand e^{-t} as a power series, evaluate term by term and confirm the legitimacy of term by term evaluation.*

Solution

$$\mathcal{L}\{e^{-t}\} = \frac{1}{s+1}$$

$$\lim_{t \to 0} F(t) = F(0) = e^{-0} = 1$$

$$\lim_{s \to \infty} sf(s) = \lim_{s \to \infty} \frac{s}{s+1} = 1.$$

This confirms the initial value theorem. The final value theorem is also confirmed as follows:-

$$\lim_{t \to \infty} F(t) = \lim_{t \to \infty} e^{-t} = 0$$

$$\lim_{s \to 0} sf(s) = \lim_{s \to 0} \frac{s}{s+1} = 0.$$

The power series expansion for e^{-t} is

$$\begin{aligned} e^{-t} &= 1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \cdots + (-1)^n \frac{t^n}{n!} \\ \mathcal{L}\{e^{-t}\} &= \frac{1}{s} - \frac{1}{s^2} + \frac{1}{s^3} - \cdots + \frac{(-1)^n}{s^{n+1}} \\ &= \frac{1}{s}\left(1 + \frac{1}{s}\right)^{-1} = \frac{1}{s+1}. \end{aligned}$$

Hence the term by term evaluation of the power series expansion for e^{-t} gives the right answer. This is not a proof of the series expansion method of course, merely a verification that the method gives the right answer in this instance.

2.6 The Impulse Function

There is a whole class of "functions" that, strictly, are not functions at all. In order to be a function, an expression has to be defined for all values of the variable in the specified range. When this is not so, then the expression is not a function because it is not well defined. It may not seem at all sensible for us to bother with such creatures, in that if a function is not defined at a certain point then what use is it? However, if a “function” instead of being well defined possesses some global property, then it indeed does turn out to be worth considering such pathological objects. Of course, having taken the decision to consider such objects, strictly there needs to be a whole new mathematical language constructed to deal with them. Notions such as adding them together, multiplying them, performing operations such as integration cannot be done without preliminary mathematics. The general consideration of this kind of object forms the study of *generalised functions* (see Jones 1966 or Lighthill 1970) which is outside the scope of this text. For our purposes we introduce the first such function which occurred naturally in the field of electrical engineering and is the so called impulse function. It is sometimes called Dirac’s δ function after the pioneering theoretical physicist P.A.M. Dirac (1902–1984). It has the following definition which involves its integral. This has not been defined properly, but if we write the definition first we can then comment on the integral.

Definition 2.3 *The Dirac-δ function $\delta(t)$ is defined as having the following properties*

$$\delta(t) = 0 \;\forall t \;, t \neq 0 \tag{2.3}$$

$$\int_{-\infty}^{\infty} h(t)\delta(t)dt = h(0) \tag{2.4}$$

for any function $h(t)$ continuous in $(-\infty, \infty)$.

We shall see in the next paragraph that the Dirac-δ function can be thought of as the limiting case of a top hat function of unit area as it becomes infinitesimally thin but infinitely tall, i.e. the following limit

$$\delta(t) = \lim_{T\to\infty} T_p(t)$$

where

$$T_p(t) = \begin{cases} 0 & t \leq -1/T \\ \frac{1}{2}T & -1/T < t < 1/T \\ 0 & t \geq 1/T. \end{cases}$$

The integral in the definition can then be written as follows:

$$\int_{-\infty}^{\infty} h(t) \lim_{T\to\infty} T_p(t)dt = \lim_{T\to\infty} \int_{-\infty}^{\infty} h(t)T_p(t)dt$$

provided the limits can be exchanged which of course depends on the behaviour of the function $h(t)$ but this can be so chosen to fulfil our needs. The integral inside the limit exists, being the product of continuous functions, and its value is the area under the curve $h(t)T_p(t)$. This area will approach the value $h(0)$ as $T \to \infty$ by the following argument. For sufficiently large values of T, the interval $[-1/T, 1/T]$

will be small enough for the value of $h(t)$ not to differ very much from its value at the origin. In this case we can write $h(t) = h(0) + \epsilon(t)$ where $|\epsilon(t)|$ is in some sense small and tends to zero as $T \to \infty$. The integral thus can be seen to tend to $h(0)$ as $T \to \infty$ and the property is established.

Returning to the definition of $\delta(t)$ strictly, the first condition is redundant; only the second is necessary, but it is very convenient to retain it. Now as we have said, $\delta(t)$ is not a true function because it has not been defined for $t = 0$. $\delta(0)$ has no value. Equivalent conditions to Eq. (2.4) are:-

$$\int_{0-}^{\infty} h(t)\delta(t)dt = h(0)$$

and

$$\int_{-\infty}^{0+} h(t)\delta(t)dt = h(0).$$

These follow from a similar argument as before using a limiting definition of $\delta(t)$ in terms of the top hat function. In this section, wherever the integral of a δ function (or later related "derivatives") occurs it will be assumed to involve this kind of limiting process. The details of taking the limit will however be omitted.

Let us now look at a more visual approach. As we have seen algebraically in the last paragraph $\delta(t)$ is sometimes called the impulse function because it can be thought of as the shape of Fig. 2.3, the top hat function if we let $T \to \infty$. Of course there are many shapes that will behave like $\delta(t)$ in some limit. The top hat function is one of the simplest to state and visualise. The crucial property is that the area under this top hat function is unity for all values of T, so letting $T \to \infty$ preserves this property. Diagrammatically, the Dirac-δ or impulse function is represented by an arrow as in Fig. 2.4 where the length of the arrow is unity. Using Eq. (2.4) with $h \equiv 1$ we see that

$$\int_{-\infty}^{\infty} \delta(t)dt = 1$$

which is consistent with the area under $\delta(t)$ being unity.

We now ask ourselves what is the Laplace transform of $\delta(t)$? Does it exist? We suspect that it might be 1 for Eq. (2.4) with $h(t) = e^{-st}$, a perfectly valid choice of $h(t)$ gives

$$\int_{-\infty}^{\infty} \delta(t)e^{-st}dt = \int_{0-}^{\infty} \delta(t)e^{-st}dt = 1.$$

However, we progress with care. This is good advice when dealing with generalised functions. Let us take the Laplace transform of the top hat function $T_p(t)$ defined mathematically by

$$T_p(t) = \begin{cases} 0 & t \leq -1/T \\ \frac{1}{2}T & -1/T < t < 1/T \\ 0 & t \geq 1/T. \end{cases}$$

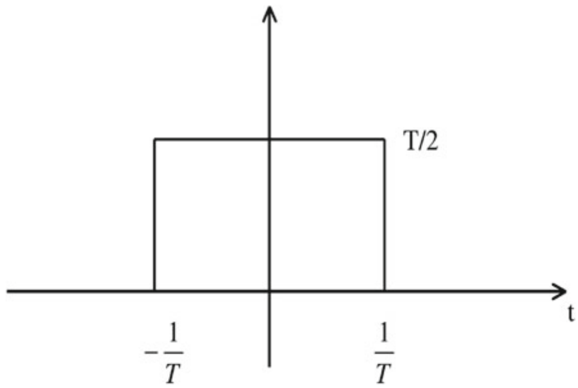


Fig. 2.3 The "top hat" function

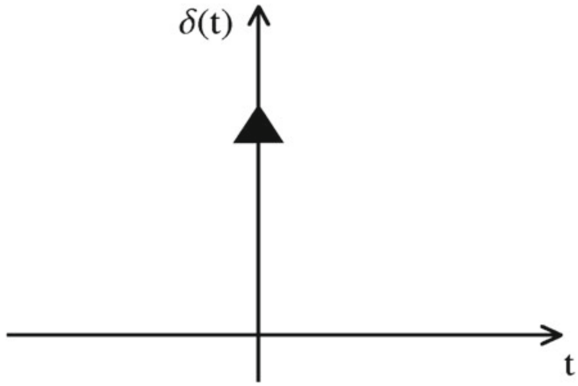


Fig. 2.4 The Dirac-δ function

The calculation proceeds as follows:-

$$\begin{aligned}\mathcal{L}\{T_p(t)\} &= \int_0^\infty T_p(t)e^{-st}dt \\ &= \int_0^{1/T} \frac{1}{2}Te^{-st}dt \\ &= \left[-\frac{T}{2s}e^{-st}\right]_0^{1/T} \\ &= \left[\frac{T}{2s} - \frac{T}{2s}e^{-s/T}\right].\end{aligned}$$

As $T \to \infty$,

$$e^{-s/T} \approx 1 - \frac{s}{T} + O\left(\frac{1}{T^2}\right)$$

hence

$$\frac{T}{2s} - \frac{T}{2s}e^{-s/T} \approx \frac{1}{2} + O\left(\frac{1}{T}\right)$$

which $\to \frac{1}{2}$ as $T \to \infty$.

In Laplace transform theory it is usual to define the impulse function $\delta(t)$ such that

$$\mathcal{L}\{\delta(t)\} = 1.$$

This means reducing the width of the top hat function so that it lies between 0 and $1/T$ (not $-1/T$ and $1/T$) and increasing the height from $\frac{1}{2}T$ to T in order to preserve unit area. Clearly the difficulty arises because the impulse function is centred on $t = 0$ which is precisely the lower limit of the integral in the definition of the Laplace transform. Using 0- as the lower limit of the integral overcomes many of the difficulties.

The function $\delta(t - t_0)$ represents an impulse that is centred on the time $t = t_0$. It can be considered to be the limit of the function $K(t)$ where $K(t)$ is the displaced top hat function defined by

$$K(t) = \begin{cases} 0 & t \le t_0 - 1/2T \\ \frac{1}{2}T & t_0 - 1/2T < t < t_0 + 1/2T \\ 0 & t \ge t_0 + 1/2T \end{cases}$$

as $T \to \infty$. The definition of the delta function can be used to deduce that

$$\int_{-\infty}^{\infty} h(t)\delta(t - t_0)dt = h(t_0)$$

and that, provided $t_0 > 0$

$$\mathcal{L}\{\delta(t - t_0)\} = e^{-st_0}.$$

Letting $t_0 \to 0$ leads to

$$\mathcal{L}\{\delta(t)\} = 1$$

a correct result. Another interesting result can be deduced almost at once and expresses mathematically the property of $\delta(t)$ to pick out a particular function value, known to engineers as the *filtering property*. Since

$$\int_{-\infty}^{\infty} h(t)\delta(t - t_0)dt = h(t_0)$$

with $h(t) = e^{-st} f(t)$ and $t_0 = a \geq 0$ we deduce that

$$\mathcal{L}\{\delta(t-a)f(t)\} = e^{-as} f(a).$$

Mathematically, the impulse function has additional interest in that it enables insight to be gained into the properties of discontinuous functions. From a practical point of view too there are a number of real phenomena that are closely approximated by the delta function. The sharp blow from a hammer, the discharge of a capacitor or even the sound of the bark of a dog are all in some sense impulses. All of this provides motivation for the study of the delta function.

One property that is particularly useful in the context of Laplace transforms is the value of the integral

$$\int_{-\infty}^{t} \delta(u - u_0)du.$$

This has the value 0 if $u_0 > t$ and the value 1 if $u_0 < t$. Thus we can write

$$\int_{-\infty}^{t} \delta(u - u_0)du = \begin{cases} 0 & 0 < u_0 \\ 1 & t > u_0 \end{cases}$$

or

$$\int_{-\infty}^{t} \delta(u - u_0)du = H(t - u_0)$$

where H is Heaviside's unit step function. If we were allowed to differentiate this result, or to put it more formally to use the fundamental theorem of the calculus (on functions one of which is not really a function, a second which is not even continuous let alone differentiable) then one could write that "$\delta(u - u_0) = H'(u - u_0)$" or state that "the impulse function is the derivative of the Heaviside unit step function". Before the pure mathematicians send out lynching parties, let us examine these loose notions. Everywhere except where $u = u_0$ the statement is equivalent to stating that the derivative of unity is zero, which is obviously true. The additional information in the albeit loose statement in quotation marks is a quantification of the nature of the unit jump in $H(u - u_0)$. We know the gradient there is infinite, but the nature of it is embodied in the second integral condition in the definition of the delta function, Eq. (2.4). The subject of *generalised functions* is introduced through this concept and the interested reader is directed towards the texts by Jones and Lighthill. All that will be noted here is that it is possible to define a whole string of derivatives $\delta'(t)$, $\delta''(t)$, etc. where all these derivatives are zero everywhere except at $t = 0$. The key to keeping rigorous here is the property

$$\int_{-\infty}^{\infty} h(t)\delta(t)dt = h(0).$$

The "derivatives" have analogous properties, viz.

$$\int_{-\infty}^{\infty} h(t)\delta'(t)dt = -h'(0)$$

and in general

$$\int_{-\infty}^{\infty} h(t)\delta^{(n)}(t)dt = (-1)^n h^{(n)}(0).$$

Of course, the function $h(t)$ will have to be appropriately differentiable. Now the Laplace transform of this nth derivative of the Dirac delta function is required. It can be easily deduced that

$$\int_{-\infty}^{\infty} e^{-st}\delta^{(n)}(t)dt = \int_{0-}^{\infty} e^{-st}\delta^{(n)}(t)dt = s^n.$$

Notice that for all these generalised functions, the condition for the validity of the initial value theorem is violated, and the final value theorem although perfectly valid is entirely useless. It is time to do a few examples.

Example 2.6 *Determine the inverse Laplace transform*

$$\mathcal{L}^{-1}\left\{\frac{s^2}{s^2+1}\right\}$$

and interpret the $F(t)$ *obtained.*

Solution Writing

$$\frac{s^2}{s^2+1} = 1 - \frac{1}{s^2+1}$$

and using the linearity property of the inverse Laplace transform gives

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{s^2}{s^2+1}\right\} &= \mathcal{L}^{-1}\{1\} - \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}\\ &= \delta(t) - \sin t.\end{aligned}$$

This function is sinusoidal with a unit impulse at $t = 0$.

Note the direct use of the inverse $\mathcal{L}^{-1}\{1\} = \delta(t)$. This arises straight away from our definition of $\mathcal{L}$. It is quite possible for other definitions of Laplace transform to give the value $\frac{1}{2}$ for $\mathcal{L}\{\delta(t)\}$ (for example). This may worry those readers of a pure mathematical bent. However, as long as there is consistency in the definitions of the delta function and the Laplace transform and hence its inverse, then no inconsistencies arise. The example given above will always yield the same answer $\mathcal{L}^{-1}\left\{\frac{s^2}{s^2+1}\right\} = \delta(t) - \sin t$. The small variations possible in the definition of the Laplace transform around $t = 0$ do not change this. Our definition, viz.

$$\mathcal{L}\{F(t)\} = \int_{0-}^{\infty} e^{-st} F(t) dt$$

remains the most usual.

Example 2.7 *Find the value of* $\mathcal{L}^{-1}\left\{\frac{s^3}{s^2+1}\right\}$.

Solution Using a similar technique to the previous example we first see that

$$\frac{s^3}{s^2+1} = s - \frac{s}{s^2+1}$$

so taking inverse Laplace transforms using the linearity property once more yields

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{s^3}{s^2+1}\right\} &= \mathcal{L}^{-1}\{s\} - \mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\}\\ &= \delta'(t) - \cos t\end{aligned}$$

where $\delta'(t)$ is the first derivative of the Dirac-δ function which was defined earlier.

Notice that the first derivative formula:

$$\mathcal{L}\{F'(t)\} = sf(s) - F(0)$$

with $F'(t) = \delta'(t) - \cos t$ gives

$$\mathcal{L}\{\delta'(t) - \cos t\} = \frac{s^3}{s^2+1} - F(0)$$

which is indeed the above result apart from the troublesome $F(0)$. $F(0)$ is of course not defined. Care indeed is required if standard Laplace transform results are to be applied to problems containing generalised functions. When in doubt, the best advice is to use limit definitions of $\delta(t)$ and the like, and follow the mathematics through carefully, especially the swapping of integrals and limits. The little book by Lighthill is full of excellent practical advice.

2.7 Periodic Functions

We begin with a very straightforward definition that should be familiar to everyone:

Definition 2.4 *If $F(t)$ is a function that obeys the rule*

$$F(t) = F(t+\tau)$$

for some real τ for all values of t then $F(t)$ is called a periodic function with period τ.

Periodic functions play a very important role in many branches of engineering and applied science, particularly physics. One only has to think of springs or alternating current present in household electricity to realise their prevalence. Here, a theorem on the Laplace transform of periodic functions is introduced, proved and used in some illustrative examples.

Theorem 2.8 *Let* $F(t)$ *have period* $T > 0$ *so that* $F(t) = F(t+T)$. *Then*

$$\mathcal{L}\{F(t)\} = \frac{\int_0^T e^{-st}F(t)dt}{1-e^{-sT}}.$$

Proof Like many proofs of properties of Laplace transforms, this one begins with its definition then evaluates the integral by using the periodicity of $F(t)$

$$\begin{aligned}\mathcal{L}\{F(t)\} &= \int_0^\infty e^{-st}F(t)dt \\ &= \int_0^T e^{-st}F(t)dt + \int_T^{2T} e^{-st}F(t)dt \\ &\quad + \int_{2T}^{3T} e^{-st}F(t)dt + \cdots + \int_{(n-1)T}^{nT} e^{-st}F(t)dt + \cdots\end{aligned}$$

provided the series on the right hand side is convergent. This is assured since the function $F(t)$ satisfies the condition for the existence of its Laplace transform by construction. Consider the integral

$$\int_{(n-1)T}^{nT} e^{-st}F(t)dt$$

and substitute $u = t - (n-1)T$. Since F has period T this leads to

$$\int_{(n-1)T}^{nT} e^{-st}F(t)dt = e^{-s(n-1)T}\int_0^T e^{-su}F(u)du \qquad n = 1, 2, \ldots$$

which gives

$$\begin{aligned}\int_0^\infty e^{-st}F(t)dt &= (1 + e^{-sT} + e^{-2sT} + \cdots)\int_0^T e^{-st}F(t)dt \\ &= \frac{\int_0^T e^{-st}F(t)dt}{1-e^{-sT}}\end{aligned}$$

on summing the geometric progression. This proves the result. □

Here is an example of using this theorem.

Example 2.8 *A rectified sine wave is defined by the expression*

$$F(t) = \begin{cases} \sin t & 0 < t < \pi \\ -\sin t & \pi < t < 2\pi \end{cases}$$
$$F(t) = F(t + 2\pi)$$

determine $\mathcal{L}\{F(t)\}$.

Solution The graph of $F(t)$ is shown in Fig. 2.5. The function $F(t)$ actually has period π, but it is easier to carry out the calculation as if the period was 2π. Additionally we can check the answer by using the theorem with $T = \pi$. With $T = 2\pi$ we have from Theorem 2.8,

$$\mathcal{L}\{F(t)\} = \frac{\int_0^{2\pi} e^{-st} F(t)dt}{1 - e^{-sT}}$$

where the integral in the numerator is evaluated by splitting into two as follows:-

$$\int_0^{2\pi} e^{-st} F(t)dt = \int_0^{\pi} e^{-st} \sin t dt + \int_{\pi}^{2\pi} e^{-st}(-\sin t)dt.$$

Now, writing $\Im\{\}$ to denote the imaginary part of the function in the brace we have

$$\begin{aligned} \int_0^{\pi} e^{-st} \sin t dt &= \Im\left\{\int_0^{\pi} e^{-st+it} dt\right\} \\ &= \Im\left[\frac{1}{i-s} e^{-st+it}\right]_0^{\pi} \\ &= \Im\left\{\frac{1}{i-s}(e^{-s\pi+i\pi} - 1)\right\} \\ &= \Im\left\{\frac{1}{s-i}(1 + e^{-s\pi})\right\}. \end{aligned}$$

So

$$\int_0^{\pi} e^{-st} \sin t dt = \frac{1 + e^{-\pi s}}{1 + s^2}.$$

Similarly,

$$\int_{\pi}^{2\pi} e^{-st} \sin t dt = -\frac{e^{-2\pi s} + e^{-\pi s}}{1 + s^2}.$$

Hence we deduce that

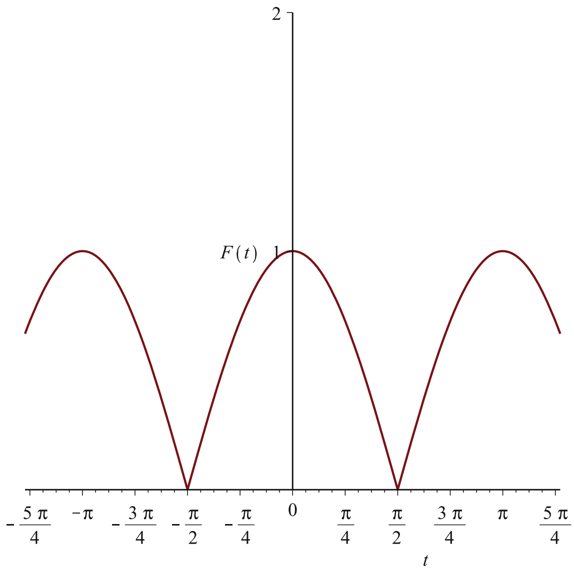


Fig. 2.5 The graph of $F(t)$

$$\mathcal{L}\{F(t)\} = \frac{(1+e^{-\pi s})^2}{(1+s^2)(1-e^{-2\pi s})}$$
$$= \frac{1+e^{-\pi s}}{(1+s^2)(1-e^{-\pi s})}.$$

This is precisely the answer that would have been obtained if Theorem 2.8 had been applied to the function

$$F(t) = \sin t \quad 0 < t < \pi \quad F(t) = F(t+\pi).$$

We can therefore have some confidence in our answer.

2.8 Exercises

1. If $F(t) = \cos(at)$, use the derivative formula to re-establish the Laplace transform of $\sin(at)$.
2. Use Theorem 2.1 with
$$F(t) = \int_0^t \frac{\sin u}{u} du$$
to establish the result.

$$\mathcal{L}\left\{\frac{\sin(at)}{t}\right\} = \tan^{-1}\left\{\frac{a}{s}\right\}.$$

3. Prove that
$$\mathcal{L}\left\{\int_0^t \int_0^v F(u)dudv\right\} = \frac{f(s)}{s^2}.$$

4. Find
$$\mathcal{L}\left\{\int_0^t \frac{\cos(au) - \cos(bu)}{u} du\right\}.$$

5. Determine
$$\mathcal{L}\left\{\frac{2\sin t \sinh t}{t}\right\}.$$

6. Prove that if $\bar{f}(s)$ indicates the Laplace transform of a piecewise continuous function $f(t)$ then
$$\lim_{s\to\infty} \bar{f}(s) = 0.$$

7. Determine the following inverse Laplace transforms by using partial fractions
$$\text{(a)}\ \frac{2(2s+7)}{(s+4)(s+2)}, \quad s > -2 \ \text{(b)}\ \frac{s+9}{s^2-9},$$
$$\text{(c)}\ \frac{s^2+2k^2}{s(s^2+4k^2)}, \qquad \text{(d)}\ \frac{1}{s(s+3)^2},$$
$$\text{(e)}\ \frac{1}{(s-2)^2(s+3)^3}.$$

8. Verify the initial value theorem, for the two functions
(a) $2 + \cos t$ and
(b) $(4 + t)^2$.
9. Verify the final value theorem, for the two functions
(a) $3 + e^{-t}$ and
(b) $t^3 e^{-t}$.
10. Given that
$$\mathcal{L}\{\sin(\sqrt{t})\} = \frac{k}{s^{3/2}} e^{-1/4s}$$
use $\sin x \sim x$ near $x = 0$ to determine the value of the constant k. (You will need the table of standard transforms Appendix B.)
11. By using a power series expansion, determine (in series form) the Laplace transforms of $\sin(t^2)$ and $\cos(t^2)$.
12. $P(s)$ and $Q(s)$ are polynomials, the degree of $P(s)$ is less than that of $Q(s)$ which is n. Use partial fractions to prove the result

$$\mathcal{L}^{-1}\left\{\frac{P(s)}{Q(s)}\right\} = \sum_{k=1}^{n} \frac{P(\alpha_k)}{Q'(\alpha_k)} e^{\alpha_k t}$$

where α_k are the n distinct zeros of $Q(s)$.

13. Find the following Laplace transforms:

(a)

$$H(t-a)$$

(b)

$$f_1 = \begin{cases} t+1 & 0 \le t \le 2 \\ 3 & t > 2 \end{cases}$$

(c)

$$f_2 = \begin{cases} t+1 & 0 \le t \le 2 \\ 6 & t > 2 \end{cases}$$

(d) the derivative of $f_1(t)$.

14. Find the Laplace transform of the triangular wave function:

$$F(t) = \begin{cases} t & 0 \le t < c \\ 2c - t & c \le t < 2c \end{cases}$$

$$F(t+2c) = F(t).$$

15. Find the Laplace transform of the generally placed top hat function:

$$F(t) = \begin{cases} \frac{1}{h} & a \le t < a+h \\ 0 & \text{otherwise} \end{cases}.$$

Hence deduce the Laplace transform of the Dirac-δ function $\delta(t-a)$ where $a > 0$ is a real constant.

Chapter 3
Convolution and the Solution of Ordinary Differential Equations

3.1 Introduction

It is assumed from the outset that students will have some familiarity with ordinary differential equations (ODEs), but there is a brief résumé given in Sect. 3.3. The other central and probably new idea is that of the convolution integral and this is introduced fully in Sect. 3.2. Of course it is possible to solve some kinds of differential equation without using convolution as is obvious from the last chapter, but mastery of the convolution theorem greatly extends the power of Laplace transforms to solve ODEs. In fact, familiarity with the convolution operation is necessary for the understanding of many other topics such as the solution of partial differential equations (PDEs) and those that are outside the scope of this book such as the use of Green's functions for forming the general solution of various types of boundary value problem (BVP).

3.2 Convolution

The definition of convolution is straightforward.

Definition 3.1 *The convolution of two given functions $f(t)$ and $g(t)$ is written $f * g$ and is defined by the integral*

$$f * g = \int_0^t f(\tau)g(t-\tau)d\tau.$$

The only condition that is necessary to impose on the functions f and g is that their behaviour be such that the integral on the right exists. Piecewise continuity of both in the interval $[0, t]$ is certainly sufficient. The following definition of piecewise continuity is repeated here for convenience.

Definition 3.2 *If an interval* $[0, t_0]$ *say can be partitioned into a finite number of subintervals* $[0, t_1], [t_1, t_2], [t_2, t_3], \ldots, [t_n, t_0]$ *with* $0, t_1, t_2, \ldots, t_n, t_0$ *an increasing*

P. Dyke, *An Introduction to Laplace Transforms and Fourier Series*,
Springer Undergraduate Mathematics Series, DOI: 10.1007/978-1-4471-6395-4_3,

sequence of times and such that a given function $f(t)$ *is continuous in each of these subintervals but not necessarily at the end points themselves, then* $f(t)$ *is piecewise continuous in the interval* $[0, t_0]$.

It is easy to prove the following theorem

Theorem 3.1 **(Symmetry)** $f * g = g * f$.

It is left as an exercise to the student to prove this.

Probably the most important theorem concerning the use of Laplace transforms and convolution is introduced now. It is called the convolution theorem and enables one, amongst other things, to deduce the inverse Laplace transform of an expression provided it can be expressed in the form of a product of functions, each inverse Laplace transform of which is known. Thus, in a loose sense, the inverse Laplace transform is equivalent to integration by parts, although unlike integration by parts, there is no integration left to do on the right hand side.

Theorem 3.2 **(Convolution)** *If* $f(t)$ *and* $g(t)$ *are two functions of exponential order (so that their Laplace transforms exist), and writing* $\mathcal{L}\{f\} = \bar{f}(s)$ *and* $\mathcal{L}\{g\} = \bar{g}(s)$ *as the two Laplace transforms then* $\mathcal{L}^{-1}\{\bar{f}\bar{g}\} = f * g$ *where* $*$ *is the convolution operator introduced above.*

Proof In order to prove this theorem, we in fact show that

$$\bar{f}\bar{g} = \mathcal{L}\{f(t) * g(t)\}$$

by direct integration of the right hand side. In turn, this involves its interpretation in terms of a repeated integral. Now,

$$\mathcal{L}\{f(t) * g(t)\} = \int_0^\infty e^{-st} \int_0^t f(\tau)g(t-\tau)d\tau dt$$

using the definition of the Laplace transform. The domain of this repeated integral takes the form of a wedge in the t, τ plane. This wedge (infinite wedge) is displayed in Fig. 3.1. Trivial rewriting of this double integral to facilitate changing the order of integration gives

$$\mathcal{L}\{f(t) * g(t)\} = \int_0^\infty \int_0^t e^{-st} f(\tau)g(t-\tau)d\tau dt$$

and thus integrating with respect to t first (horizontally first instead of vertically in Fig. 3.1) gives

$$\begin{aligned}\mathcal{L}\{f(t) * g(t)\} &= \int_0^\infty \int_\tau^\infty e^{-st} f(\tau)g(t-\tau)dt d\tau \\ &= \int_0^\infty f(\tau)\left\{\int_\tau^\infty e^{-st} g(t-\tau)dt\right\} d\tau.\end{aligned}$$

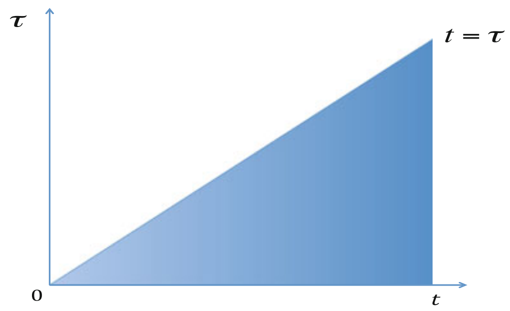


Fig. 3.1 The domain of the repeated integral

Implement the change of variable $u = t - \tau$ in the inner integral so that it becomes

$$\begin{aligned}\int_\tau^\infty e^{-st} g(t-\tau)dt &= \int_0^\infty e^{-s(u+\tau)} g(u)du \\ &= e^{-st} \int_0^\infty e^{-su} g(u)du \\ &= e^{-s\tau} \bar{g}(s).\end{aligned}$$

Thus we have

$$\begin{aligned}\mathcal{L}\{f(t) * g(t)\} &= \int_0^\infty f(\tau) e^{-s\tau} \bar{g}(s) d\tau \\ &= \bar{g}(s) \bar{f}(s) \\ &= \bar{f}(s) \bar{g}(s).\end{aligned}$$

Hence

$$f(t) * g(t) = \mathcal{L}^{-1}\{\bar{f}\bar{g}\}.$$

This establishes the theorem. □

This particular result is sometimes referred to as Borel's Theorem, and the convolution referred to as Faltung. These names are found in older books and some present day engineering texts. Before going on to use this theorem, let us do an example or two on calculating convolutions to get a feel of how the operation works.

Example 3.1 *Find the value of* $\cos t * \sin t$.

Solution Using the definition of convolution we have

$$\cos t * \sin t = \int_0^t \cos(\tau) \sin(t - \tau) d\tau.$$

To evaluate this we could of course resort to computer algebra: alternatively we use the identity

$$\sin(A)\cos(B) = \frac{1}{2}[\sin(A+B) + \sin(A-B)].$$

This identity is engraved on the brains of those who passed exams before the advent of formula sheets. Evaluating the integral by hand thus progresses as follows. Let $A = t - \tau$ and $B = \tau$ in this trigonometric formula to obtain

$$\sin(t-\tau)\cos(\tau) = \frac{1}{2}[\sin t + \sin(t - 2\tau)]$$

whence,

$$\begin{aligned}
\cos t * \sin t &= \int_0^t \cos\tau \sin(t-\tau)d\tau \\
&= \frac{1}{2}\int_0^t [\sin t + \sin(t-2\tau)]d\tau \\
&= \frac{1}{2}\sin t\,[\tau]_0^t + \frac{1}{4}\,[\cos(t-2\tau)]_0^t \\
&= \frac{1}{2}t\sin t + \frac{1}{4}\,[\cos(-t) - \cos t] \\
&= \frac{1}{2}t\sin t.
\end{aligned}$$

Let us try another example.

Example 3.2 *Find the value of* $\sin t * t^2$.

Solution We progress as before by using the definition

$$\sin t * t^2 = \int_0^t (\sin\tau)(t-\tau)^2 d\tau.$$

It is up to us to choose the order as from Theorem 3.2 $f * g = g * f$. Of course we choose the order that gives the easier integral to evaluate. In fact there is little to choose in this present example, but it is a point worth watching in future. This integral is evaluated by integration by parts. Here are the details.

$$\begin{aligned}
\sin t * t^2 &= \int_0^t (\sin\tau)(t-\tau)^2 d\tau \\
&= \Big[-(t-\tau)^2\cos\tau\Big]_0^t - \int_0^t 2(t-\tau)\cos\tau d\tau \\
&= t^2 - 2\left\{[(t-\tau)\sin\tau]_0^t - \int_0^t \sin\tau d\tau\right\}
\end{aligned}$$

$$= t^2 - 2\left\{0 + [-\cos\tau]_0^t\right\}$$
$$= t^2 + 2\cos t - 2.$$

Of course the integration can be done by computer algebra.

Both of these examples provide typical evaluations of convolution integrals. Convolution integrals occur in many different branches of engineering, particularly when signals are being processed. However, we shall only be concerned with their application to the evaluation of the inverse Laplace transform. Therefore without further ado, let us do a couple of these examples.

Example 3.3 *Find the following inverse Laplace transforms:*

$$(a)\ \mathcal{L}^{-1}\left\{\frac{s}{(s^2+1)^2}\right\},$$

$$(b)\ \mathcal{L}^{-1}\left\{\frac{1}{s^3(s^2+1)}\right\}.$$

Solution (a) We cannot evaluate this inverse Laplace transform in any direct fashion. However we do know the standard forms

$$\mathcal{L}^{-1}\left\{\frac{s}{(s^2+1)}\right\} = \cos t \quad \text{and} \quad \mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)}\right\} = \sin t.$$

Hence

$$\mathcal{L}\{\cos t\}\mathcal{L}\{\sin t\} = \frac{s}{(s^2+1)^2}$$

and so using the convolution theorem, and Example 3.1

$$\mathcal{L}^{-1}\left\{\frac{s}{(s^2+1)^2}\right\} = \cos t * \sin t = \frac{1}{2}t\sin t.$$

(b) Proceeding similarly with this inverse Laplace transform, we identify the standard forms:-

$$\mathcal{L}^{-1}\left\{\frac{1}{s^3}\right\} = \frac{1}{2}t^2 \quad \text{and} \quad \mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)}\right\} = \sin t.$$

Thus

$$\mathcal{L}\{t^2\}\mathcal{L}\{\sin t\} = \frac{2}{s^3(s^2+1)}$$

and so

$$\mathcal{L}^{-1}\left\{\frac{1}{s^3(s^2+1)}\right\} = \frac{1}{2}t^2 * \sin t.$$

This convolution has been found in Example 3.2, hence the inverse Laplace transform is

$$\mathcal{L}^{-1}\left\{\frac{1}{s^3(s^2+1)}\right\} = \frac{1}{2}(t^2 + 2\cos t - 2).$$

In this case there is an alternative approach as the expression

$$\frac{1}{s^3(s^2+1)}$$

can be decomposed into partial fractions and the inverse Laplace transform evaluated by that method.

In a sense, the last example was "cooked" in that the required convolutions just happened to be those already evaluated. Nevertheless the power of the convolution integral is clearly demonstrated. In the kind of examples met here there is usually little doubt that the functions meet the conditions necessary for the existence of the Laplace transform. In the real world, the functions may be time series, have discontinuities or exhibit a stochastic character that makes formal checking of these conditions awkward. It remains important to do this checking however: that is the role of mathematics. Note that the product of two functions that are of exponential order is also of exponential order and the integral of this product is of exponential order too.

The next step is to use the convolution theorem on more complicated results. To do this requires the derivation of the "well known" integral

$$\int_0^\infty e^{-t^2}dt = \frac{1}{2}\sqrt{\pi}.$$

Example 3.4 *Use a suitable double integral to evaluate the improper integral*

$$\int_0^\infty e^{-t^2}dt.$$

Solution Consider the double integral

$$\int\int_S e^{-(x^2+y^2)}dS$$

where S is the quarter disc $x \geq 0, y \geq 0,\ \ x^2 + y^2 \leq a^2$. Converting to polar co-ordinates (R, θ) this integral becomes

$$\int_0^a \int_0^{\frac{\pi}{2}} e^{-R^2} R d\theta dR$$

where $R^2 = x^2 + y^2,\ R\cos\theta = x,\ R\sin\theta = y$ so that $dS = Rd\theta dR$. Evaluating this integral we obtain

$$I = \frac{\pi}{2}\int_0^a Re^{-R^2}dR = \frac{\pi}{2}\left[-\frac{1}{2}e^{-R^2}\right]_0^a = \frac{\pi}{4}\left\{1 - e^{-a^2}\right\}.$$

As $a \to \infty,\ I \to \frac{\pi}{4}$.

We now consider the double integral

$$I_k = \int_0^k \int_0^k e^{-(x^2+y^2)}dxdy.$$

The domain of this integral is a square of side k. Now

$$I_k = \left\{\int_0^k e^{-x^2}dx\right\}\left\{\int_0^k e^{-y^2}dy\right\} = \left\{\int_0^k e^{-x^2}dx\right\}^2$$

A glance at Fig. 3.2 will show that

$$I_{a/\sqrt{2}} < I < I_a.$$

However, we can see that

$$I_k \to \left\{\int_0^\infty e^{-x^2}dx\right\}^2$$

as $k \to \infty$. Hence if we let $a \to \infty$ in the inequality

$$I_{a/\sqrt{2}} < I < I_a$$

we deduce that

$$I \to \left\{\int_0^\infty e^{-x^2}dx\right\}^2$$

as $a \to \infty$. Therefore

$$\left\{\int_0^\infty e^{-x^2}dx\right\}^2 = \frac{\pi}{4}$$

or

$$\int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$$

as required. This formula plays a central role in statistics, being one half of the area under the bell shaped curve usually associated with the normal distribution. In this text, its frequent appearance in solutions to problems involving diffusion and conduction is more relevant.

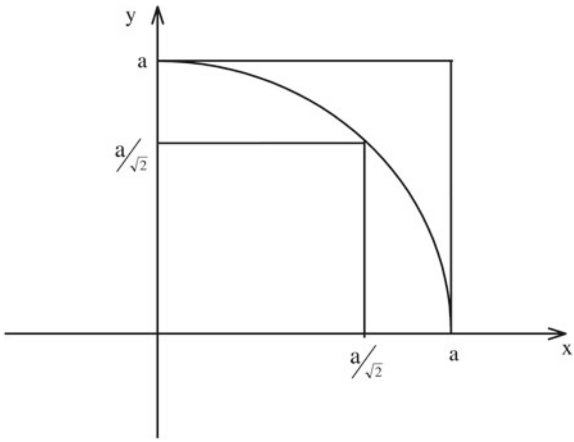


Fig. 3.2 The domains of the repeated integrals

Let us use this integral to find the Laplace transform of $1/\sqrt{t}$. From the definition of Laplace transform, it is necessary to evaluate the integral

$$\mathcal{L}\left\{\frac{1}{\sqrt{t}}\right\} = \int_0^\infty \frac{e^{-st}}{\sqrt{t}}dt$$

provided we can be sure it exists. The behaviour of the integrand at ∞ is not in question, and the potential problem at the origin disappears once it is realised that $1/\sqrt{t}$ itself is integrable in any finite interval that contains the origin. We can therefore proceed to evaluate it. This is achieved through the substitution $st = u^2$. The conversion to the variable u results in

$$\int_0^\infty \frac{e^{-st}}{\sqrt{t}}dt = \int_0^\infty \frac{e^{-u^2}}{s}\frac{\sqrt{s}}{u}2udu = \frac{2}{\sqrt{s}}\int_0^\infty e^{-u^2}du = \sqrt{\frac{\pi}{s}}.$$

We have therefore established the Laplace transform result:

$$\mathcal{L}\left\{\frac{1}{\sqrt{t}}\right\} = \sqrt{\frac{\pi}{s}}$$

and, perhaps more importantly, the inverse

$$\mathcal{L}^{-1}\left\{\frac{1}{\sqrt{s}}\right\} = \frac{1}{\sqrt{\pi t}}.$$

These results have wide application. Their use with the convolution theorem opens up a whole new class of functions on which the Laplace transform and its inverse can operate. The next example is typical.

Example 3.5 *Determine*

$$\mathcal{L}^{-1}\left\{\frac{1}{\sqrt{s}(s-1)}\right\}.$$

Solution The only sensible way to proceed using our present knowledge is to use the results

$$\mathcal{L}^{-1}\left\{\frac{1}{\sqrt{s}}\right\} = \frac{1}{\sqrt{\pi t}}$$

and (by the shifting property)

$$\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} = e^t.$$

Whence, using the convolution theorem

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{1}{\sqrt{s}(s-1)}\right\} &= \int_0^t \frac{1}{\sqrt{\pi\tau}} e^{(t-\tau)} d\tau \\ &= \frac{e^t}{\sqrt{\pi}} \int_0^t \frac{e^{-\tau}}{\sqrt{\tau}} d\tau.\end{aligned}$$

The last integral is similar to that evaluated in determining the Laplace transform of $1/\sqrt{t}$ except for the upper limit. It is tackled using the same transformation, only this time we keep careful track of the upper limit. Therefore, substitute $\tau = u^2$ to obtain

$$\int_0^t \frac{e^{-\tau}}{\sqrt{\tau}} d\tau = 2\int_0^{\sqrt{t}} e^{-u^2} du.$$

Now, this integral is well known from the definition of the Error function, $\text{erf}(x)$.

Definition 3.3 *The Error Function erf(x) is defined by*

$$erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt.$$

It is related to the area under the normal distribution curve in statistics. The factor of $2/\sqrt{\pi}$ is there to ensure that the total area is unity as is required by probability theory. Returning to Laplace transforms, it is thus possible to express the solution as follows:-

$$\mathcal{L}^{-1}\left\{\frac{1}{\sqrt{s}(s-1)}\right\} = e^t \text{erf}\{\sqrt{t}\}.$$

The function $1 - \text{erf}(x)$ is called the complementary error function and is written $\text{erfc}(x)$. It follows immediately that

$$\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-t^2} dt \quad \text{and} \quad \operatorname{erf}(x) + \operatorname{erfc}(x) = 1.$$

It is in fact the complementary error function rather than the error function itself that emerges from solving the differential equations related to diffusion problems. These problems are met explicitly in Chap. 5, but here we solve the problem of finding the Laplace transform of the function

$$t^{-3/2} \exp\left\{-\frac{k^2}{4t}\right\}$$

where k is a constant representing the diffusion called the diffusion coefficient. Here it can be thought of as an unspecified constant.

Example 3.6 *Determine*

$$\mathcal{L}\left\{t^{-3/2} \exp\left\{-\frac{k^2}{4t}\right\}\right\}.$$

Solution We start as always by writing the Laplace transform explicitly in terms of the integral definition, and it looks daunting. To evaluate it does require some convoluted algebra. As with all algebraic manipulation these days, the option is there to use computer algebra, although existing systems would find this hard. In any case, the derivation of this particular formula by hand does help in the understanding of its subsequent use. So, we start with

$$\mathcal{L}\left\{t^{-3/2} \exp\left\{-\frac{k^2}{4t}\right\}\right\} = \int_0^{\infty} e^{-k^2/4t} e^{-st} t^{-3/2} dt.$$

First of all, let us substitute $u = k/2\sqrt{t}$. This is done so that a term e^{-u^2} appears in the integrand. Much other stuff appears too of course and we now sort this out:

$$du = -\frac{k}{4} t^{-3/2} dt$$

which eliminates the $t^{-3/2}$ term. The limits swap round, then swap back again once the negative sign from the du term is taken into account. Thus we obtain

$$\int_0^{\infty} e^{-k^2/4t} e^{-st} t^{-3/2} dt = \frac{4}{k} \int_0^{\infty} e^{-u^2} e^{-sk/4u^2} du$$

and this completes the first stage. We now strive to get the integral on the right into error function form. (Computer algebra systems need leading by the nose through this kind of algebra.) First of all we complete the square

$$u^2 + \frac{sk^2}{4u^2} = \left\{u - \frac{k\sqrt{s}}{2u}\right\}^2 + k\sqrt{s}.$$

The unsquared term is independent of u; this is important. The integral can thus be written:

$$\int_0^\infty e^{-k^2/4t} e^{-st} t^{-3/2} dt = \frac{4}{k} e^{-k\sqrt{s}} \int_0^\infty e^{-\left(u - \frac{k\sqrt{s}}{2u}\right)^2} du.$$

This completes the second stage. The third and perhaps most bizarre stage of evaluating this integral involves consideration and manipulation of the integral

$$\int_0^\infty e^{-(u - \frac{a}{u})^2} du, \text{ where } a = \frac{k\sqrt{s}}{2}.$$

If we let $v = a/u$ in this integral, then since

$$\left(u - \frac{a}{u}\right)^2 = \left(v - \frac{a}{v}\right)^2$$

but $du = -adv/v^2$ we obtain the unexpected result

$$\int_0^\infty e^{-(u - \frac{a}{u})^2} du = \int_0^\infty \frac{a}{u^2} e^{-(u - \frac{a}{u})^2} du.$$

The minus sign cancels with the exchange in limits as before, and the dummy variable v has been replaced by u. We can use this result to deduce immediately that

$$\int_0^\infty \left(1 + \frac{a}{u^2}\right) e^{-(u - \frac{a}{u})^2} du = 2\int_0^\infty e^{-(u - \frac{a}{u})^2} du.$$

In the left hand integral, we substitute $\lambda = u - a/u$ so that $d\lambda = (1 + a/u^2)du$. In this way, we regain our friend from Example 3.4, apart from the lower limit which is $-\infty$ rather than 0. Finally therefore

$$\begin{aligned}\int_0^\infty \left(1 + \frac{a}{u^2}\right) e^{-(u - \frac{a}{u})^2} du &= \int_{-\infty}^\infty e^{-\lambda^2} d\lambda \\ &= 2\int_0^\infty e^{-\lambda^2} d\lambda \\ &= \sqrt{\pi}.\end{aligned}$$

Hence we have deduced that

$$\int_0^\infty e^{-(u - \frac{a}{u})^2} du = \frac{1}{2}\sqrt{\pi}$$

and is independent of the constant a. Using these results, a summary of the calculation of the required Laplace transform is

$$\begin{aligned}\mathcal{L}\left\{t^{-3/2}\exp\left\{-\frac{k^2}{4t}\right\}\right\} &= \frac{4}{k}e^{-k\sqrt{s}}\int_0^\infty e^{(u-\frac{k\sqrt{s}}{2u})^2}du.\\ &= \frac{4}{k}e^{-k\sqrt{s}}\frac{1}{2}\sqrt{\pi}\\ &= \frac{2\sqrt{\pi}}{k}e^{-k\sqrt{s}}.\end{aligned}$$

Taking the inverse Laplace transform of this result gives the equally useful formula

$$\mathcal{L}^{-1}\left\{e^{-k\sqrt{s}}\right\} = \frac{k}{2\sqrt{\pi t^3}}e^{-k^2/4t}.$$

As mentioned earlier, this Laplace transform occurs in diffusion and conduction problems. In particular for the applied mathematician, it enables the estimation of possible time scales for the diffusion of pollutant from a point source. Let us do one more example using the result just derived.

Example 3.7 *Use the convolution theorem to find*

$$\mathcal{L}^{-1}\left\{\frac{e^{-k\sqrt{s}}}{s}\right\}.$$

Solution We note the result just derived, namely

$$\mathcal{L}^{-1}\left\{e^{-k\sqrt{s}}\right\} = \frac{k}{2\sqrt{\pi t^3}}e^{-k^2/4t}.$$

together with the standard result

$$\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = 1$$

to deduce that

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{e^{-k\sqrt{s}}}{s}\right\} &= \frac{k}{2\sqrt{\pi t^3}}e^{-k^2/4t} * 1\\ &= \frac{k}{2\sqrt{\pi}}\int_0^t \tau^{-3/2}e^{-k^2/4\tau}d\tau.\end{aligned}$$

We evaluate this integral by the (by now familiar) trick of substituting $u^2 = k^2/4\tau$. This means that

$$2udu = -\frac{k^2}{4\tau^2}d\tau$$

and the limits transform from $\tau = 0$ and $\tau = t$ to $u = \infty$ and $u = k/2\sqrt{t}$ respectively. They swap round due to the negative sign in the expression for du so we obtain

$$\begin{aligned}\frac{k}{2\sqrt{\pi}}\int_0^t \tau^{-3/2}e^{-k^2/4\tau}d\tau &= \frac{2}{\sqrt{\pi}}\int_{k/2\sqrt{t}}^{\infty} e^{-u^2}du \\ &= \text{erfc}\left(\frac{k}{2\sqrt{t}}\right).\end{aligned}$$

Hence we have the result

$$\mathcal{L}^{-1}\left\{\frac{e^{-k\sqrt{s}}}{s}\right\} = \text{erfc}\left(\frac{k}{2\sqrt{t}}\right)$$

which is also of some significance in the modelling of diffusion. An alternative derivation of this result not using convolution is possible using a result from Chap. 2, viz.

$$\mathcal{L}^{-1}\left\{\frac{\bar{f}(s)}{s}\right\} = \int_0^t f(u)du.$$

This formula can be regarded as a special case of the convolution theorem. We shall make further use of the convolution theorem in this kind of problem in Chap. 6. In the remainder of this chapter we shall apply the results of Chap. 2 to the solution of ordinary differential equations (ODEs). This also makes use of the convolution theorem both as an alternative to using partial fractions but more importantly to enable general solutions to be written down explicitly even where the right hand side of the ODE is a general function.

3.3 Ordinary Differential Equations

At the outset we stress that all the functions in this section will be assumed to be appropriately differentiable. For the examples in this section which are algebraically explicit this is obvious, but outside this section and indeed outside this text care needs to be taken to ensure that this remains the case. It is of course a stricter criterion than that needed for the existence of the Laplace transform (that the function be of exponential order) so using the Laplace transform as a tool for solving ordinary differential equations is usually not a problem. On the other hand, using differential equations to establish results for Laplace transforms is certainly to be avoided as this automatically imposes the strict condition of differentiability on the functions in them. It is perhaps the premier aim of mathematics to remove restrictions and widen the class of functions that obey theorems and results, not the other way round.

Most of you will be familiar to a greater or lesser extent with differential equations: however for completeness a résumé is now given of the basic essentials. A differential equation is an equation where the unknown is in the form of a derivative. Operating on a derivative with the Laplace transform can eliminate the derivative, replacing each differentiation with a multiple of s. It should not be surprising therefore that Laplace transforms are a handy tool for solving certain types of differential equation. Before going into detail, let us review some of the general terminology met in discussing differential equations.

The *order* of an ordinary differential equation is the highest derivative attained by the unknown. Thus the equation

$$\left(\frac{dy}{dx}\right)^3 + y = \sin x$$

is a first order equation. The equation

$$\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^4 + \ln x = 0$$

is, on the other hand a second order equation. The equation

$$\left(\frac{d^3y}{dx^3}\right)^4 + \left(\frac{dy}{dx}\right)^7 + y^8 = 0$$

is of third order. Such exotic equations will not (cannot) be solved using Laplace transforms. Instead we shall be restricted to solving first and second order equations which are linear. Linearity has already been met in the context of the Laplace transform. The linearity property is defined by

$$L\{\alpha y_1 + \beta y_2\} = \alpha L\{y_1\} + \beta L\{y_2\}.$$

In a linear differential equation, the dependent variable obeys this linearity property. We shall only be considering linear differential equations here (Laplace transforms being linear themselves are only useful for solving linear differential equations). Differential equations that *cannot* be solved are those containing powers of the unknown or expressions such as $\tan(y)$, e^y. Thus we will solve first and second order linear differential equations. Although this seems rather restrictive, it does account for nearly all those linear ODEs found in real life situations. The word *ordinary* denotes that there is only differentiation with respect to a single independent variable so that the solution is the required function of this variable. If the number of variables is two or more, the differential equation becomes a *partial* differential equation (PDE) and these are considered later in Chap. 5.

The Laplace transform of a derivative was found easily by direct integration by parts. The two useful results that will be used extensively here are

$$\mathcal{L}\{f'(t)\} = s\bar{f}(s) - f(0)$$

and

$$\mathcal{L}\{f''(t)\} = s^2\bar{f}(s) - sf(0) - f'(0)$$

where the prime denotes differentiation with respect to t. Note that the right hand sides of both of these expressions involve knowledge of $f(t)$ at $t = 0$. This is important. Also, the order of the derivative determines how many arbitrary constants the solution contains. There is one arbitrary constant for each integration, so a first order ODE will have one arbitrary constant, a second order ODE two arbitrary constants, etc. There are complications over uniqueness with differential equations that are not linear, but fortunately this does not concern us here. We know that the Laplace transform is a linear operator; it is not easily applied to non-linear problems.

Upon taking Laplace transforms of a linear ODE, the derivatives themselves disappear, transforming into the Laplace transform of the function multiplied by s (for a first derivative) or s^2 (for a second derivative). Moreover, the correct number of constants also appear in the form of $f(0)$ (for a first order ODE) and $f(0)$ and $f'(0)$ (for a second order ODE). Some texts conclude therefore that Laplace transforms can be used only to solve *initial value problems*, that is problems where enough information is known at the start to solve it. This is not strictly true. Whilst it remains the case that initial value problems are best suited to this method of solution, two point boundary value problems can be solved by transforming the equation, and retaining $f(0)$ and $f'(0)$ (or the first only) as unknowns. These unknowns are then found algebraically by the substitution of the given boundary conditions and the solving of the resulting differential equation. We shall, however almost always be solving ODEs with initial conditions (but see Example 3.9). From a physical standpoint this is entirely reasonable. The Laplace transform is a mapping from t space to s space and t almost always corresponds to time. For problems involving time, the situation is known *now* and the equation(s) are solved in order to determine what is going on *later*. This is indeed the classical initial value problem. We are now ready to try a few examples.

Example 3.8 *Solve the first order differential equation*

$$\frac{dx}{dt} + 3x = 0 \textit{ where } x(0) = 1.$$

Solution Note that we have abandoned $f(t)$ for the more usual $x(t)$, but this should be regarded as a trivial change of dummy variable. This rather simple differential equation can in fact be solved by a variety of methods. Of course we use Laplace transforms, but it is useful to check the answer by solving again using separation of variables or integrating factor methods as these will be familiar to most students.

Taking Laplace transforms leads to

$$\mathcal{L}\left\{\frac{dx}{dt}\right\} + 3\mathcal{L}\{x\} = 0$$

which implies

$$s\bar{x}(s) - x(0) + 3\bar{x}(s) = 0$$

using the standard overbar to denote Laplace transform. Since $x(0) = 1$, solving for $\bar{x}(s)$ gives

$$\bar{x}(s) = \frac{1}{s+3}$$

whence

$$x(t) = \mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\} = e^{-3t}$$

using the standard form. That this is indeed the solution is easily checked.

Let us look at the same equation, but with a different boundary condition.

Example 3.9 *Solve the first order differential equation*

$$\frac{dx}{dt} + 3x = 0 \textit{ where } x(1) = 1.$$

Solution Proceeding as before, we now cannot insert the value of $x(0)$ so we arrive at the solution

$$x(t) = \mathcal{L}^{-1}\left\{\frac{x(0)}{s+3}\right\} = x(0)e^{-3t}.$$

We now use the boundary condition we do have to give

$$x(1) = x(0)e^{-3} = 1$$

which implies

$$x(0) = e^3$$

and the solution is

$$x(t) = e^{3(1-t)}.$$

Here is a slightly more challenging problem.

Example 3.10 *Solve the differential equation*

$$\frac{dx}{dt} + 3x = \cos 3t \textit{ given } x(0) = 0.$$

Solution Taking the Laplace transform (we have already done this in Example 3.8 for the left hand side) we obtain

$$s\bar{x}(s) - x(0) + 3\bar{x}(s) = \frac{s}{s^2 + 9}$$

using standard forms. With the zero initial condition solving this for $\bar{x}(s)$ yields

$$\bar{x}(s) = \frac{s}{(s+3)(s^2+9)}.$$

This solution is in the form of the product of two known Laplace transforms. Thus we invert either using partial fractions or the convolution theorem: we choose the latter. First of all note the standard forms

$$\mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\} = e^{-3t} \text{ and } \mathcal{L}^{-1}\left\{\frac{s}{s^2+9}\right\} = \cos(3t).$$

Using the convolution theorem yields:

$$x(t) = \mathcal{L}^{-1}\left\{\frac{1}{s+3}\frac{s}{s^2+9}\right\} = \int_0^t e^{-3(t-\tau)}\cos(3\tau)d\tau.$$

(The equally valid choice of

$$\int_0^t e^{-3\tau}\cos\left(3(t-\tau)\right)\,d\tau$$

could have been made, but as a general rule it is better to arrange the order of the convolution so that the $(t-\tau)$ is in an exponential if you have one.) The integral is straightforward to evaluate using integration by parts or computer algebra. The gory details are omitted here. The result is

$$\int_0^t e^{-3\tau}\cos(3\tau)d\tau = \frac{1}{6}(e^{3t}\cos(3t) + e^{3t}\sin(3t) - 1).$$

Thus we have

$$\begin{aligned} x(t) &= e^{-3t}\int_0^t e^{-3\tau}\cos(3\tau)d\tau \\ &= \frac{1}{6}(\cos(3t) + \sin(3t)) - \frac{1}{6}e^{-3t}. \end{aligned}$$

This solution could have been obtained by partial fractions which is algebraically simpler. It is also possible to solve the original ODE by complementary function/particular integral techniques or to use integrating factor methods. The choice is yours. In the next example there is a clear winner. It is also possible to get a closed form answer using integrating factor techniques, but using Laplace transforms together with the convolution theorem is our choice here.

Example 3.11 *Find the general solution to the differential equation*

$$\frac{dx}{dt} + 3x = f(t) \text{ where } x(0) = 0,$$

and $f(t)$ is of exponential order (which is sufficient for the method of solution used to be valid).

Solution It is compulsory to use convolution here as the right hand side is an arbitrary function. The Laplace transform of the equation leads directly to

$$\begin{aligned} x(t) &= \mathcal{L}^{-1}\left\{\frac{\bar{f}(s)}{s+3}\right\} \\ &= \int_0^t e^{-3(t-\tau)} f(\tau)d\tau \end{aligned}$$

so that

$$x(t) = e^{-3t}\int_0^t e^{3\tau} f(\tau)d\tau.$$

The function $f(t)$ is of course free to be assigned. In engineering and other applied subjects, $f(t)$ is made to take exotic forms; the discrete numbers corresponding to the output of laboratory measurements perhaps or even a time series with a stochastic (probabilistic) nature. However, here $f(t)$ must comply with our basic definition of a function. The ability to solve this kind of differential equation even with the definition of function met here has important practical consequences for the engineer and applied scientist. The function $f(t)$ is termed *input* and the term $x(t)$ *output*. To get from one to the other needs a *transfer function*. In the last example, the function $1/(s+3)$ written in terms of the transform variable s is this transfer function. This is the language of systems analysis, and such concepts also form the cornerstone of control engineering. They are also vital ingredients to branches of electrical engineering and the machine dynamics side of mechanical engineering. In mathematics, the procedure for writing the solution to a non-homogeneous differential equation (that is one with a non-zero right hand side) in terms of the solution of the corresponding homogeneous differential equation involves the development of the complementary function and particular solution. Complementary functions and particular solutions are standard concepts in solving second order ordinary differential equations, the subject of the next section.

3.3.1 Second Order Differential Equations

Let us now do a few examples to see how Laplace transforms are used to solve *second* order ordinary differential equations. The technique is no different from solving first order ODEs, but finding the inverse Laplace transform is often more challenging. Let us start by finding the solution to a homogeneous second order ODE that will be familiar to most of you who know about oscillations.

Example 3.12 *Use Laplace transforms to solve the equation*

$$\frac{d^2x}{dt^2} + x = 0 \textit{ with } x(0) = 1,\ x'(0) = 0.$$

Solution Taking the Laplace transform of this equation using the usual notation gives

$$s^2\bar{x}(s) - sx(0) - x'(0) + \bar{x}(s) = 0.$$

With $x(0) = 1$ and $x'(0) = 0$ we obtain

$$\bar{x}(s) = \frac{s}{s^2 + 1}.$$

This is a standard form which inverts to $x(t) = \cos t$. That this is the correct solution to this simple harmonic motion problem is easy to check.

Why not try changing the initial condition to $y(0) = 0$ and $y'(0) = 1$ which should lead to $y(t) = \sin t$? We are now ready to build on this result and solve the inhomogeneous problem that follows.

Example 3.13 *Find the solution to the differential equation*

$$\frac{d^2x}{dt^2} + x = t \textit{ with } x(0) = 1,\ x'(0) = 0.$$

Solution Apart from the trivial change of variable, we follow the last example and take Laplace transforms to obtain

$$s^2\bar{x}(s) - sx(0) - x'(0) + \bar{x} = \mathcal{L}\{t\} = \frac{1}{s^2}.$$

With start conditions $x(0) = 1$ and $x'(0) = 0$ this gives

$$\bar{x}(s)(s^2 + 1) - s = \frac{1}{s^2}$$

$$\text{so } \bar{x}(s) = \frac{s}{s^2 + 1} + \frac{1}{s^2(s^2 + 1)}.$$

Taking the inverse Laplace transform thus gives:

$$
\begin{aligned}
x &= \mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2+1)}\right\}\\
&= \cos t + \int_0^t (t-\tau)\sin(\tau)d\tau
\end{aligned}
$$

using the convolution theorem. Integrating by parts (omitting the details) gives

$$x = \cos t - \sin t + t.$$

The first two terms are the complementary function and the third the particular integral. The whole is easily checked to be the correct solution. It is up to the reader to decide whether this approach to solving this particular differential equation is any easier than the alternatives. The Laplace transform method provides the solution of the differential equation with a *general* right hand side in a simple and straightforward manner.

However, instead of restricting attention to this particular second order differential equation, let us consider the more general equation

$$a\frac{d^2x}{dt^2} + b\frac{dx}{dt} + cx = f(t) \quad (t \geq 0)$$

where a, b and c are constants. We will not solve this equation, but discuss it in the context of applications.

In engineering texts these constants are given names that have engineering significance. Although this text is primarily for a mathematical audience, it is nevertheless useful to run through these terms. In mechanics, a is the *mass*, b is the *damping constant* (diagrammatically represented by a dashpot), c is the *spring constant* (or *stiffness*) and x itself is the displacement of the mass. In electrical circuits, a is the *inductance*, b is the *resistance*, c is the reciprocal of the capacitance sometimes called the *reactance* and x (replaced by q) is the charge, the rate of change of which with respect to time is the more familiar electric current. Some of these names will be encountered later when we do applied examples. The right-hand side is called the *forcing* or *excitation*. In terms of systems engineering, $f(t)$ is the system input, and $x(t)$ is the system output. Since a, b and c are all constant the system described by the equation is termed *linear* and *time invariant*. It will seem very odd to a mathematician to describe a system governed by a time-dependent differential equation as "time invariant" but this is standard engineering terminology.

Taking the Laplace transform of this general second order differential equation, assuming all the appropriate conditions hold of course, yields

$$a(s^2\bar{x}(s) - sx(0) - x'(0)) + b(s\bar{x}(s) - x(0)) + c\bar{x}(s) = \bar{f}(s).$$

It is normally not a problem to assume that $\bar{x}(s)$ and $\bar{f}(s)$ are of exponential order, but just occasionally when problems have a stochastic or numerical input, care needs to be taken. Making $\bar{x}(s)$ the subject of this equation gives

$$\bar{x}(s) = \frac{\bar{f}(s) + (as+b)x(0) + ax'(0)}{as^2 + bs + c}.$$

Hence, in theory, $x(t)$ can be found by taking inverse Laplace transform. The simplest case to consider is when $x(0)$ and $x'(0)$ are both zero. The output is then free from any embellishments that might be there because of special start conditions. In this special case,

$$\bar{x}(s) = \frac{1}{as^2 + bs + c}\bar{f}(s).$$

This equation is starkly in the form "response = transfer function $\times$ input" which makes it very clear why Laplace transforms are highly regarded by engineers. The formula for $\bar{x}(s)$ can be inverted using the convolution theorem and examples of this can be found later. First however let us solve a few simpler second order differential equations explicitly.

Example 3.14 *Use Laplace transform techniques to find the solution to the second order differential equation*

$$\frac{d^2x}{dt^2} + 5\frac{dx}{dt} + 6x = 2e^{-t} \quad t \geq 0,$$

subject to the conditions $x = 1$ *and* $x' = 0$ *at* $t = 0$.

Solution Taking the Laplace transform of this equation we obtain using the usual overbar notation,

$$s^2\bar{x}(s) - sx(0) - x'(0) + 5(s\bar{x}(s) - x(0)) + 6\bar{x}(s) = \mathcal{L}\{e^{-t}\} = \frac{2}{s+1}$$

where the standard Laplace transform

$$\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$$

for any constant a has been used. Inserting the initial conditions $x = 1$ and $x' = 0$ at $t = 0$ and rearranging the formula as an equation for $\bar{x}(s)$ gives

$$(s^2 + 5s + 6)\bar{x}(s) = \frac{2}{s+1} + s + 5.$$

Factorising gives

$$\bar{x}(s) = \frac{2}{(s+1)(s+2)(s+3)} + \frac{s+5}{(s+2)(s+3)}.$$

There are many ways of inverting this expression, the easiest being to use the partial fraction method. Doing this but omitting the details gives:

$$\bar{x}(s) = \frac{1}{s+1} + \frac{1}{s+2} - \frac{1}{s+3}.$$

This inverts immediately to

$$x(t) = e^{-t} + e^{-2t} - e^{-3t} \qquad (t \geq 0).$$

The first term is the particular integral (or particular solution) and the last two terms the complementary function. The whole solution is overdamped and therefore non-oscillatory: this is undeniably the easiest case to solve as it involves very little algebra. However, it is also physically the least interesting as the solution dies away to zero very quickly. It does serve to demonstrate the power of the Laplace transform technique to solve this kind of ordinary differential equation.

An obvious question to ask at this juncture is how is it known whether a particular inverse Laplace transform can be found? We know of course that it is obtainable in principle, but this is a practical question. In Chap. 8 we derive a general form for the inverse that helps to answer this question. Only a brief and informal answer can be given at this stage. As long as the function $\bar{f}(s)$ has a finite number of finite isolated singularities then inversion can go ahead. If $\bar{f}(s)$ does not tend to zero for large $|s|$ generalised functions are to be expected, and if $\bar{f}(s)$ has a square root or a more elaborate multi-valued nature then direct inversion is made more complicated, although there is no formal difficulty. In this case, error functions, Bessel functions and the like usually feature in the solution. Most of the time, solving second order linear differential equations is straightforward and involves no more than elementary transcendental functions (exponential and trigonometric functions).

The next problem is more interesting from a physical point of view.

Example 3.15 *Use Laplace transforms to solve the following ordinary differential equation*

$$\frac{d^2x}{dt^2} + 6\frac{dx}{dt} + 9x = \sin t \quad (t \geq 0),$$

subject to $x(0) = 0$ *and* $x'(0) = 0$.

Solution With no right hand side, and with zero initial conditions, $x(t) = 0$ would result. However with the sinusoidal forcing, the solution turns out to be quite interesting. The formal way of tackling the problem is the same as for any second order differential equation with constant coefficients. Thus the mathematics follows that of the last example. Taking Laplace transforms, the equation becomes

$$s^2\bar{x}(s) - sx(0) - x'(0) + 6s\bar{x}(s) - 6x(0) + 9\bar{x}(s) = \frac{1}{s^2+1}$$

this time not lingering over the standard form (for $\mathcal{L}\{\sin(t)\}$). With the boundary conditions inserted and a little tidying this becomes

$$\bar{x}(s) = \frac{1}{s^2+1} \cdot \frac{1}{(s+3)^2}.$$

Once again either partial fractions or convolution can be used. Convolution is our choice this time. Note that

$$\mathcal{L}\{\sin t\} = \frac{1}{s^2+1} \text{ and } \mathcal{L}\{te^{-3t}\} = \frac{1}{(s+3)^2}$$

so

$$\mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)(s+3)^2}\right\} = \int_0^t \tau e^{-3\tau}\sin(t-\tau)d\tau.$$

This integral yields to integration by parts several times (or computer algebra, once). The result follows from application of the formula:

$$\begin{aligned}\int_0^t \tau e^{-3\tau}\sin(t-\tau)d\tau = &-\frac{1}{10}\int_0^t e^{-3\tau}\cos(t-\tau)d\tau \\ &+ \frac{3}{10}\int_0^t e^{-3\tau}\sin(t-\tau)d\tau + \frac{1}{10}te^{-3t}.\end{aligned}$$

However we omit the details. The result is

$$\begin{aligned}x(t) &= \mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)(s+3)^2}\right\} \\ &= \int_0^t \tau e^{-3\tau}\sin(t-\tau)d\tau \\ &= \frac{e^{-3t}}{50}(5t+3) - \frac{3}{50}\cos t + \frac{2}{25}\sin t.\end{aligned}$$

The first term is the particular solution (called the *transient* response by engineers since it dies away for large times), and the final two terms the complementary function (rather misleadingly called the *steady state* response by engineers since it persists. Of course there is nothing steady about it). After a "long time" has elapsed, the response is harmonic at the same frequency as the forcing frequency. The "long time" is in fact in practice quite short as is apparent from the graph of the output $x(t)$ which is displayed in Fig. 3.3. The graph is indistinguishable from a sinusoid after about

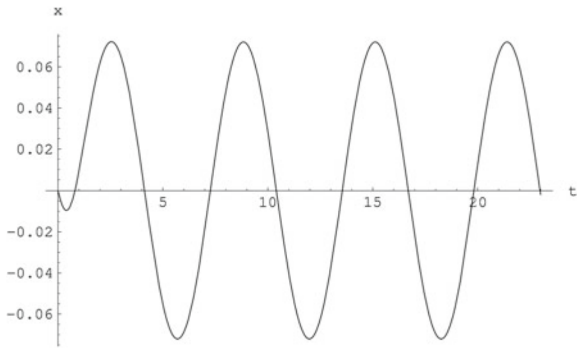


Fig. 3.3 The graph of $x(t)$

$t = 0.5$. However the amplitude and phase of the resulting oscillations are different. In fact, the combination

$$-\frac{3}{50}\cos t + \frac{2}{25}\sin t = \frac{1}{10}\sin(t - \epsilon)$$

puts the steady state into amplitude and phase form. $\frac{1}{10}$ is the amplitude and ϵ is the phase ($\cos(\epsilon) = \frac{4}{5}$, $\sin(\epsilon) = \frac{3}{5}$ for this solution). It is the fact that the response frequency is the same as the forcing frequency that is important in practical applications.

There is very little more to be said in terms of mathematics about the solution to second order differential equations with constant coefficients. The solutions are oscillatory, decaying, a mixture of the two, oscillatory and growing or simply growing exponentially. The forcing excites the response and if the response is at the same frequency as the natural frequency of the differential equation, resonance occurs. This leads to enhanced amplitudes at these frequencies. If there is no damping, then resonance leads to infinite amplitude response. Further details about the properties of the solution of second order differential equations with constant coefficients can be found in specialist books on differential equations and would be out of place here. What follows are examples where the power of the Laplace transform technique is clearly demonstrated in terms of solving practical engineering problems. It is at the very applied end of applied mathematics.

In the following example, a problem in electrical circuits is solved. As mentioned in the preamble to Example 3.14 the constants in a linear differential equation can be given significance in terms of the basic elements of an electrical circuit: *resistors, capacitors* and *inductors*. Resistors have resistance R measured in ohms, capacitors have capacitance C measured in farads, and inductors have inductance L measured in henrys. A current j flows through the circuit and the current is related to the charge q by

$$j = \frac{dq}{dt}.$$

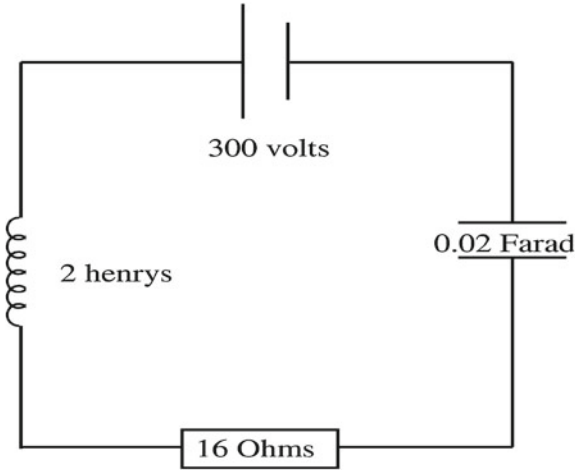


Fig. 3.4 The simple circuit

The laws obeyed by a (passive) electrical circuit are:

1. *Ohm's law* whereby the voltage drop across a resistor is Rj.
2. The voltage drop across an inductor is

$$L\frac{dj}{dt}.$$

3. The voltage drop across a capacitor is

$$\frac{q}{C}.$$

Hence in terms of q the voltage drops are respectively

$$R\frac{dq}{dt},\quad L\frac{d^2q}{dt^2} \text{ and } \frac{q}{C}$$

which enables the circuit laws (Kirchhoff's Laws) to be expressed in terms of differential equations of second order with constant coefficients (L, R and $1/C$). The forcing function (input) on the right hand side is supplied by a voltage source, e.g. a battery. Here is a typical example.

Example 3.16 *Find the differential equation obeyed by the charge for the simple circuit shown in Fig.* 3.4*, and solve it by the use of Laplace transforms given* $j = 0,\quad q = 0$ *at* $t = 0$.

Solution The current is j and the charge is q, so the voltage drop across the three devices are

$$2\frac{dj}{dt} = 2\frac{d^2q}{dt^2},\quad 16j = 16\frac{dq}{dt},\ \text{and}\ \frac{q}{0.002} = 50q.$$

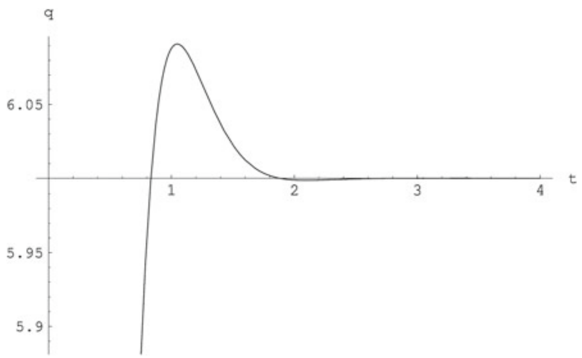


Fig. 3.5 The solution $q(t)$; $q(0) = 0$ although true, is not shown due to the fine scale on the q axis

This must be equal to 300 (the voltage output of the battery), hence

$$2\frac{d^2q}{dt^2} + 16\frac{dq}{dt} + 50q = 300.$$

So, solving this by division by two and taking the Laplace transform results in

$$s^2\bar{q}(s) - sq(0) - q'(0) + 8s\bar{q}(s) - 8q(0) + 25\bar{q}(s) = \frac{150}{s}.$$

Imposing the initial conditions gives the following equation for $\bar{q}(s)$, the Laplace transform of $q(t)$:

$$\bar{q}(s) = \frac{150}{s(s^2 + 8s + 25)} = \frac{150}{s((s+4)^2 + 9)}.$$

This can be decomposed by using partial fractions or the convolution theorem. The former is easier. This gives:

$$\bar{q}(s) = \frac{6}{s} - \frac{6(s+4)}{((s+4)^2 + 9)} - \frac{24}{((s+4)^2 + 9)}.$$

The right hand side is now a standard form, so inversion gives:

$$q(t) = 6 - 6e^{-4t}\cos(3t) - 8e^{-4t}\sin(3t).$$

This solution is displayed in Fig. 3.5. It can be seen that the oscillations are completely swamped by the exponential decay term. In fact, the current is the derivative of this which is:-

$$j = 50e^{-4t}\sin(3t)$$

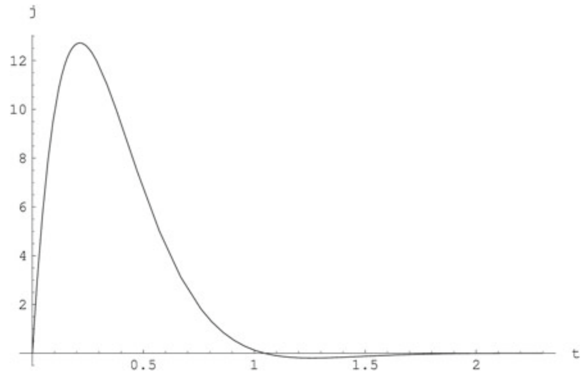


Fig. 3.6 The variation of current with time

and this is shown in Fig. 3.6 as decaying to zero very quickly. This is obviously not typical, as demonstrated by the next example. Here, we have a sinusoidal voltage source which might be thought of as mimicking the production of an alternating current.

Example 3.17 *Solve the same problem as in the previous example except that the battery is replaced by the oscillatory voltage source* $100\sin(3t)$.

Solution The differential equation is derived as before, except for the different right hand side. The equation is

$$2\frac{d^2q}{dt^2}+16\frac{dq}{dt}+50q=100\sin(3t).$$

Taking the Laplace transform of this using the zero initial conditions $q(0)=0, q'(0)=0$ proceeds as before, and the equation for the Laplace transform of $q(t)$ ($\bar{q}(s)$) is

$$\bar{q}(s)=\frac{150}{(s^2+9)((s+4)^2+9)}.$$

Note the appearance of the Laplace transform of $100\sin(3t)$ here. The choice is to either use partial fractions or convolution to invert, this time we use convolution, and this operation by its very nature recreates $100\sin(3t)$ under an integral sign "convoluted" with the complementary function of the differential equation. Recognising the two standard forms:-

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+9}\right\}=\frac{1}{3}\sin(3t) \text{ and } \mathcal{L}^{-1}\left\{\frac{1}{(s+4)^2+9}\right\}=\frac{1}{3}e^{-4t}\sin(3t),$$

gives immediately

$$q(t)=\frac{50}{3}\int_0^t e^{-4(t-\tau)}\sin(3(t-\tau))\sin(3\tau)d\tau.$$

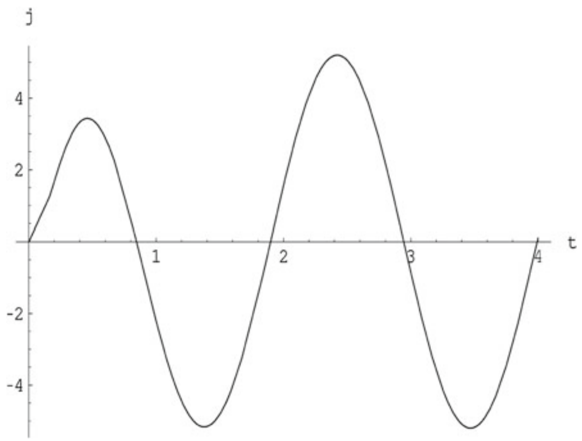


Fig. 3.7 The variation of current with time

Using integration by parts but omitting the details gives

$$j = \frac{dq(t)}{dt} = \frac{75}{52}(2\cos(3t) + 3\sin(3t)) - \frac{25}{52}e^{-4t}(17\sin(3t) + 6\cos(3t)).$$

What this solution tells the electrical engineer is that the response quickly becomes sinusoidal at the same frequency as the forcing function but with smaller amplitude and different phase. This is backed up by glancing at Fig. 3.7 which displays this solution. The behaviour of this solution is very similar to that of the mechanical engineering example, Example 3.20, that we will soon meet and demonstrates beautifully the merits of a mathematical treatment of the basic equations of engineering. A mathematical treatment enables analogies to be drawn between seemingly disparate branches of engineering.

3.3.2 Simultaneous Differential Equations

In the same way that Laplace transforms convert a single differential equation into a single algebraic equation, so they can also convert a pair of differential equations into simultaneous algebraic equations. The differential equations we solve are all linear, so a pair of linear differential equations will convert into a pair of simultaneous linear algebraic equations familiar from school. Of course, these equations will contain s, the transform variable as a parameter. These expressions in s can get quite complicated. This is particularly so if the forcing functions on the right-hand side lead to algebraically involved functions of s. Comments on the ability or otherwise of inverting these expressions remain the same as for a single differential equation. They are more complicated, but still routine. Let us start with a straightforward example.

Example 3.18 *Solve the simultaneous differential equations*

$$\frac{dx}{dt} = 2x - 3y, \; \frac{dy}{dt} = y - 2x,$$

where $x(0) = 8$ *and* $y(0) = 3$.

Solution Taking Laplace transforms and inserting the boundary conditions straight away gives:-

$$\begin{aligned} s\bar{x}(s) - 8 &= 2\bar{x}(s) - 3\bar{y}(s) \\ s\bar{y}(s) - 3 &= \bar{y}(s) - 2\bar{x}(s). \end{aligned}$$

Whence, rearranging we solve:-

$$\begin{aligned} (s-2)\bar{x} + 3\bar{y} &= 8 \\ \text{and} \quad 2\bar{x} + (s-1)\bar{y} &= 3 \end{aligned}$$

by the usual means. Using Cramer's rule or eliminating by hand gives the solutions

$$\bar{x}(s) = \frac{8s - 17}{s^2 - 3s - 4}, \; \bar{y}(s) = \frac{3s - 22}{s^2 - 3s - 4}.$$

To invert these we factorise and decompose into partial fractions to give

$$\begin{aligned} \bar{x}(s) &= \frac{5}{s+1} + \frac{3}{s-4} \\ \bar{y}(s) &= \frac{5}{s+1} - \frac{2}{s-4}. \end{aligned}$$

These invert easily and we obtain the solution

$$\begin{aligned} x(t) &= 5e^{-t} + 3e^{4t} \\ y(t) &= 5e^{-t} - 2e^{4t}. \end{aligned}$$

Even if one or both of these equations were second order, the solution method by Laplace transforms remains the same. The following example hints at how involved the algebra can get, even in the most innocent looking pair of equations.

Example 3.19 *Solve the simultaneous differential equations*

$$\begin{aligned} \frac{d^2x}{dt^2} + \frac{dy}{dt} + 3x &= 15e^{-t} \\ \frac{d^2y}{dt^2} - 4\frac{dx}{dt} + 3y &= 15\sin(2t) \end{aligned}$$

where $x = 35$, $x' = -48$, $y = 27$ *and* $y' = -55$ *at time* $t = 0$.

Solution Taking Laplace transforms of both equations as before, retaining the standard notation for the transformed variable gives

$$s^2\bar{x} - 35s + 48 + s\bar{y} - 27 + 3\bar{x} = \frac{15}{s+1}$$
$$\text{and } s^2\bar{y} - 27s + 55 - 4(s\bar{x} - 35) + 3\bar{y} = \frac{30}{s^2+4}.$$

Solving for $\bar{x}$ and $\bar{y}$ is indeed messy but routine. We do one step to group the terms together as follows:

$$(s^2+3)\bar{x} + s\bar{y} = 35s - 21 + \frac{15}{s+1}$$
$$-4s\bar{x} + (s^2+3)\bar{y} = 27s - 195 + \frac{30}{s^2+4}.$$

This time, this author has used a computer algebra package to solve these two equations. A partial fraction routine has also been used. The result is

$$\bar{x}(s) = \frac{30s}{s^2+1} - \frac{45}{s^2+9} + \frac{3}{s+1} + \frac{2s}{s^2+4}$$
$$\text{and } \bar{y}(s) = \frac{30s}{s^2+9} - \frac{60}{s^2+1} - \frac{3}{s+1} + \frac{2}{s^2+4}.$$

Inverting using standard forms gives

$$x(t) = 30\cos t - 15\sin(3t) + 3e^{-t} + 2\cos(2t)$$
$$y(t) = 30\cos(3t) - 60\sin t - 3e^{-t} + \sin(2t).$$

The last two terms on the right-hand side of the expression for both x and y resemble the forcing terms whilst the first two are in a sense the "complementary function" for the system. The motion is quite a complex one and is displayed as Fig. 3.8

Having looked at the application of Laplace transforms to electrical circuits, now let us apply them to mechanical systems. Again it is emphasised that there is no new mathematics here; it is however new *applied* mathematics.

In mechanical systems, we use Newton's second law to determine the motion of a mass which is subject to a number of forces. The kind of system best suited to Laplace transforms are the mass-spring-damper systems. Newton's second law is of the form

$$F = m\frac{d^2x}{dt^2}$$

where F is the force, m is the mass and x the displacement. The components of the system that also act on the mass m are a *spring* and a *damper*. Both of these give rise

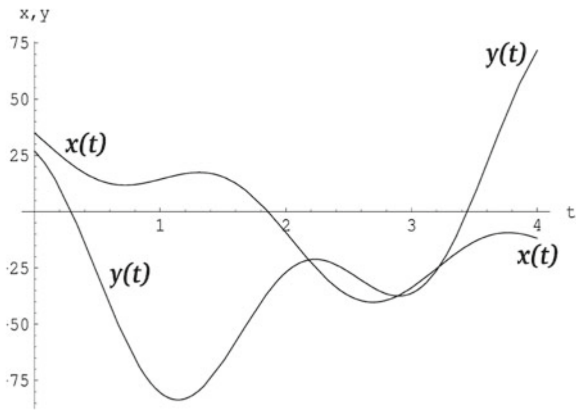


Fig. 3.8 The solutions $x(t)$ and $y(t)$

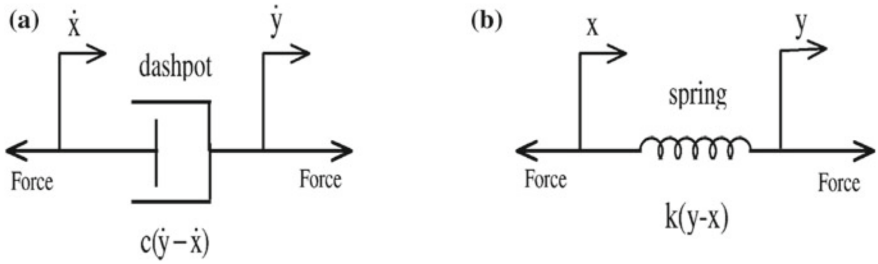


Fig. 3.9 The forces due to **a** a damper and **b** a spring

to changes in displacement according to the following rules (see Fig. 3.9). A damper produces a force proportional to the net speed of the mass but always opposes the motion, i.e. $c(\dot{y} - \dot{x})$ where c is a constant and the dot denotes differentiation with respect to t. A spring produces a force which is proportional to displacement. Here, springs will be well behaved and assumed to obey Hooke's Law. This force is $k(y-x)$ where k is a constant sometimes called the *stiffness* by mechanical engineers. To put flesh on these bones, let us solve a typical mass spring damping problem. Choosing to consider two masses gives us the opportunity to look at an application of simultaneous differential equations.

Example 3.20 *Figure 3.10 displays a mechanical system. Find the equations of motion, and solve them given that the system is initially at rest with* $x = 1$ *and* $y = 2$.

Solution Applying Newton's Second Law of Motion successively to each mass using Hooke's Law (there are no dampers) gives:-

$$m_1\ddot{x} = k_2(y - x) - k_1 x$$
$$m_2\ddot{y} = -k_3 x - k_2(y - x).$$

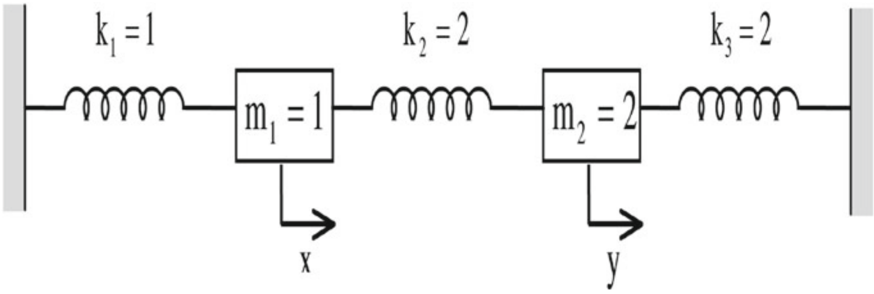


Fig. 3.10 A simple mechanical system

With the values for the constants m_1, m_2, k_1, k_2 and k_3 given in Fig. 3.10, the following differential equations are obtained:-

$$\ddot{x} + 3x - 2y = 0$$
$$2\ddot{y} + 4y - 2x = 0.$$

The mechanics (thankfully for most) is now over, and we take the Laplace transform of both equations to give:-

$$(s^2 + 3)\bar{x} - 2\bar{y} = sx(0) + \dot{x}(0)$$
$$-\bar{x} + (s^2 + 2)\bar{y} = sy(0) + \dot{y}(0).$$

The right hand side involves the initial conditions which are: $x(0) = 1$, $y(0) = 2$, $\dot{x}(0) = 0$ and $\dot{y}(0) = 0$. Solving these equations (by computer algebra or by hand) gives, for $\bar{y}$,

$$\bar{y}(s) = \frac{2s^3 + 5s}{(s^2 + 4)(s^2 + 1)} = \frac{s}{s^2 + 1} + \frac{s}{s^2 + 4}$$

and inverting gives

$$y(t) = \cos t + \cos(2t).$$

Rather than finding $\bar{x}$ and inverting, it is easier to substitute for y and its second derivative $\ddot{y}$ in the equation

$$x = \ddot{y} + 2y.$$

This finds x directly as

$$x(t) = \cos t - 2\cos(2t).$$

The solution is displayed as Fig. 3.11. It is possible and mechanically desirable to form the combinations

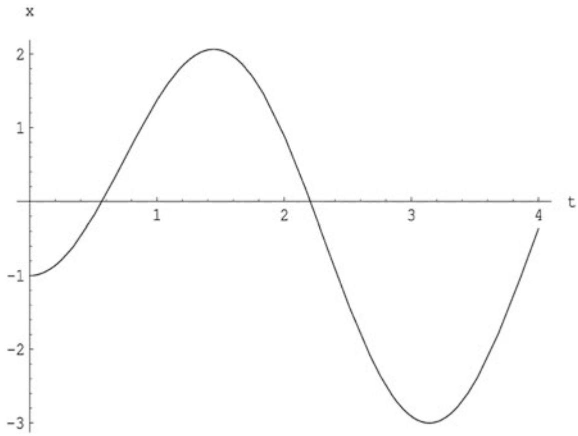


Fig. 3.11 A simple mechanical system solved

$$\frac{1}{3}(y - x) = \cos(2t) \text{ and } \frac{1}{3}(x + 2y) = \cos t$$

as these isolate the two frequencies and help in the understanding of the subsequent motion which at first sight can seem quite complex. This introduces the concept of *normal modes* which are outside the scope of this text, but very important to mechanical engineers as well as anyone else interested in the behaviour of oscillating systems.

3.4 Using Step and Impulse Functions

In Sect. 2.3 some properties of Heaviside's unit step function were explored. In this section we extend this exploration to problems that involve differential equations. As a reminder the step function $H(t)$ is defined as

$$H(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$$

and a function $f(t)$ which is switched on at time t_0 is represented simply as $f(t - t_0)H(t - t_0)$. The second shift theorem, Theorem 2.4, implies that the Laplace transform of this is given by

$$\mathcal{L}\{f(t - t_0)H(t - t_0)\} = e^{-st_0}\bar{f}(s).$$

Another useful result derived earlier is

$$\mathcal{L}\{\delta(t - a)f(t)\} = e^{-as}f(a)$$

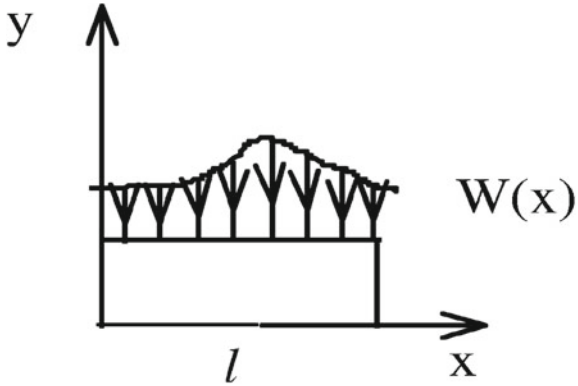


Fig. 3.12 The beam and its load $W(x)$

where $\delta(t-a)$ is the Dirac $-\delta$ or impulse function centred at $t=a$. The properties of the δ function as required in this text are outlined in Sect. 2.6. Let us use these properties to solve an engineering problem. This next example is quite extensive; more of a case study. It involves concepts usually found in mechanics texts, although it is certainly possible to solve the differential equation that is obtained abstractly and without recourse to mechanics: much would be missed in terms of realising the practical applications of Laplace transforms. Nevertheless this example can certainly be omitted from a first reading, and discarded entirely by those with no interest in applications to engineering.

Example 3.21 *The equation governing the bending of beams is of the form*

$$k\frac{d^4y}{dx^4} = -W(x)$$

where k is a constant called the flexural rigidity *(the product of Young's modulus and length in fact, but this is not important here), and $W(x)$ is the transverse force per unit length along the beam. The layout is indicated in Fig.* 3.12*. Use Laplace transforms (in x) to solve this problem and discuss the case of a point load.*

Solution There are several aspects to this problem that need a comment here. The mathematics comes down to solving an ordinary differential equation which is fourth order but easy enough to solve. In fact, only the fourth derivative of $y(x)$ is present, so in normal circumstances one might expect direct integration (four times) to be possible. That it is not is due principally to the form $W(x)$ usually takes. There is also that the beam is of finite length l. In order to use Laplace transforms the domain is extended so that $x \in [0, \infty)$ and the Heaviside step function is utilised. To progress in a step by step fashion let us consider the cantilever problem first where the beam is held at one end. Even here there are conditions imposed at the free end. However, we can take Laplace transforms in the usual way to eliminate the x derivatives. We define the Laplace transform in x as

$$\bar{y}(s) = \int_0^\infty e^{-xs} y(x) dx$$

where remember we have extended the domain to ∞. In transformed coordinates the equation for the beam becomes:-

$$k(s^4 \bar{y}(s) - s^3 y(0) - s^2 y'(0) - sy''(0) - y'''(0)) = -\overline{W}(s).$$

Thus,

$$\bar{y}(s) = \frac{-\overline{W}(s)}{ks^4} + \frac{y(0)}{s} + \frac{y'(0)}{s^2} + \frac{y''(0)}{s^3} + \frac{y'''(0)}{s^4}$$

and the solution can be found by inversion. It is at this point that the engineer would be happy, but the mathematician should be pausing for thought. The beam may be long, but it is not infinite. This being the case, is it legitimate to define the Laplace transform in x as has been done here? What needs to be done is some tidying up using Heaviside's step function. If we replace $y(x)$ by the combination $y(x)[1 - H(x - l)]$ then this latter function will certainly fulfil the necessary and sufficient conditions for the existence of the Laplace transform provided $y(x)$ is piecewise continuous. One therefore interprets $\bar{y}(s)$ as

$$\bar{y}(s) = \int_0^\infty y(x)[1 - H(x - l)]e^{-xs} dx$$

and inversion using the above equation for $\bar{y}(s)$ follows once the forcing is known. In general, the convolution theorem is particularly useful here as $W(x)$ may take the form of data (from a strain gauge perhaps) or have a stochastic character. Using the convolution theorem, we have

$$\mathcal{L}^{-1}\left\{\frac{\overline{W}(s)}{s^4}\right\} = \frac{1}{6}\int_0^x (x - \xi)^3 W(\xi) d\xi.$$

The solution to the problem is therefore

$$\begin{aligned} y(x)[1 - H(x - l)] = &-\frac{1}{6k}\int_0^x (x - \xi)^3 W(\xi) d\xi \\ &+ y(0) + xy'(0) + \frac{1}{2}x^2 y''(0) + \frac{1}{6}x^3 y'''(0). \end{aligned}$$

If the beam is freely supported at both ends, this is mechanics code for the following four boundary conditions

$$y = 0 \text{ at } x = 0,\ l \text{ (no displacement at the ends)}$$

and

$$y'' = 0 \text{ at } x = 0,\ l \text{ (no force at the ends).}$$

This enables the four constants of integration to be found. Straightforwardly, $y(0) = 0$ and $y''(0) = 0$ so

$$y(x)[1 - H(x-l)] = -\frac{1}{6k}\int_0^x (x-\xi)^3 W(\xi)d\xi + xy'(0) + \frac{1}{6}x^3 y'''(0).$$

The application of boundary conditions at $x = l$ is less easy. One method would be to differentiate the above expression with respect to x twice, but it is unclear how to apply this to the product on the left, particularly at $x = l$. The following procedure is recommended. Put $u(x) = y''(x)$ and the original differential equation, now in terms of $u(x)$ becomes

$$k\frac{d^2u}{dx^2} = -W(x)$$

with solution (obtained by using Laplace transforms as before) given by

$$u(x)[1 - H(x-l)] = -\frac{1}{k}\int_0^x (x-\xi)W(\xi)d\xi + u(0) + xu'(0).$$

Hence the following expression for $y''(x)$ has been derived

$$y''(x)[1 - H(x-l)] = -\frac{1}{6}\int_0^x (x-\xi)W(\xi)d\xi + y''(0) + xy'''(0).$$

This is in fact the result that would have been obtained by differentiating the expression for $y(x)$ twice ignoring derivatives of $[1 - H(x-l)]$. Applying the boundary conditions at $x = l$ ($y(l) = y''(l) = 0$) now gives the results

$$y'''(0) = \frac{1}{kl}\int_0^l (l-\xi)W(\xi)d\xi$$

and

$$y'(0) = \frac{1}{6kl}\int_0^l (l-\xi)^3 W(\xi)d\xi - \frac{l}{6k}\int_0^l (l-\xi)W(\xi)d\xi.$$

This provides the general solution to the problem in terms of integrals

$$\begin{aligned} y(x)[1 - H(x-l)] = &-\frac{1}{6k}\int_0^x (x-\xi)^3 W(\xi)d\xi \\ &+ \frac{1}{6kl}\int_0^l x(l-\xi)(\xi^2 - 2l\xi + x^2)W(\xi)d\xi. \end{aligned}$$

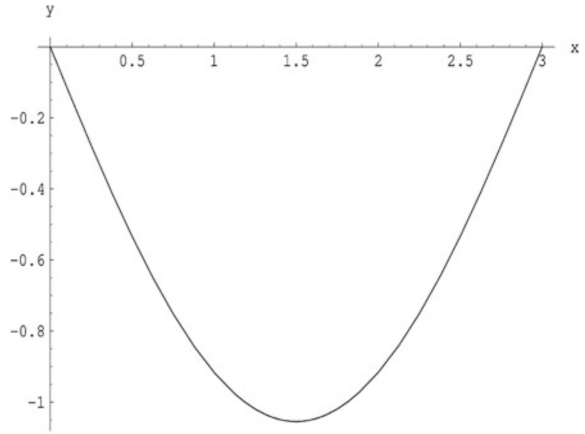


Fig. 3.13 The displacement of the beam $y(x)$ with $W = 1$ and $k = 1$. The length l equals 3

It is now possible to insert any loading function into this expression and calculate the displacement caused. In particular, let us choose the values $l = 3$, $k = 1$ and consider a uniform loading of unit magnitude $W = \text{constant} = 1$. The integrals are easily calculated in this simple case and the resulting displacement is

$$y(x) = -\frac{x^4}{24} + \frac{x^3}{4} - \frac{9x}{8}$$

which is shown in Fig. 3.13.

We now consider a more realistic loading, that of a point load located at $x = l/3$ (one third of the way along the bar). It is now the point at which the laws of mechanics need to be applied in order to translate this into a specific form for $W(x)$. This however is not a mechanics text, therefore it is quite likely that you are not familiar with enough of these laws to follow the derivation. For those with mechanics knowledge, we assume that the weight of the beam is concentrated at its mid-point ($x = l/2$) and that the beam itself is static so that there is no turning about the free end at $x = l$. If the point load has magnitude P, then the expression for $W(x)$ is

$$W(x) = \frac{W}{l}H(x) + P\delta\left(x - \frac{1}{3}l\right) - \left(\frac{1}{2}W + \frac{2}{3}P\right)\delta(x).$$

From a mathematical point of view, the interesting point here is the presence of the Dirac-δ function on the right hand side which means that integrals have to be handled with some care. For this reason, and in order to present a different way of solving the problem but still using Laplace transforms we go back to the fourth order ordinary differential equation for $y(x)$ and take Laplace transforms. The Laplace transform of the right hand side ($W(x)$) is

$$\overline{W}(s) = \frac{W}{ls} + Pe^{-\frac{1}{3}sl} - \left(\frac{1}{2}W + \frac{2}{3}P\right).$$

The boundary conditions are $y = 0$ at $x = 0,\ l$ (no displacement at the ends) and $y''(0) = 0$ at $x = 0,\ l$ (no forces at the ends). This gives

$$\bar{y}(s) = \frac{1}{k}\left[\frac{W}{ls^5} + \frac{P}{s^4}e^{-\frac{1}{3}ls} - \frac{1}{s^4}\left(\frac{1}{2}W + \frac{2}{3}P\right)\right] + \frac{y'(0)}{s^2} + \frac{y'''(0)}{s^4}.$$

This can be inverted easily using standard forms, together with the second shift theorem for the exponential term to give:-

$$\begin{aligned} y(x)[1 - H(x-l)] = -\frac{1}{k}&\left[\frac{W}{24l}x^4 + \frac{1}{6}P\left(x - \frac{1}{3}l\right)^3 - \frac{1}{6}\left(\frac{1}{2}W + \frac{2}{3}P\right)x^3\right] \\ &+ y'(0)x + \frac{1}{6}y'''(0)x^3. \end{aligned}$$

Differentiating twice gives

$$y''(x) = -\frac{1}{k}\left[\frac{1}{2l}Wx^2 + P\left(x - \frac{1}{3}l\right) - \left(\frac{1}{2}W + \frac{2}{3}P\right)x\right] + y'''(0)x, \quad 0 \le x \le l.$$

This is zero at $x = l$, whence $y'''(0) = 0$. The boundary condition $y(l) = 0$ is messier to apply as it is unnatural for Laplace transforms. It gives

$$y'(0) = -\frac{l^2}{k}\left(\frac{1}{24}W + \frac{5}{81}P\right)$$

so the solution valid for $0 \le x \le l$ is

$$\begin{aligned} y(x) = -\frac{W}{k}&\left(\frac{1}{24l}x^4 - \frac{1}{12}x^3 + \frac{1}{24}l^2x\right) - \frac{P}{k}\left(\frac{5}{81}l^2x - \frac{1}{9}x^3\right) \\ &- \frac{P}{6k}\left(x - \frac{1}{3}l\right)^3 H\left(x - \frac{1}{3}l\right). \end{aligned}$$

This solution is illustrated in Fig. 3.14. Other applications to problems that give rise to partial differential equations will have to wait until Chap. 5.

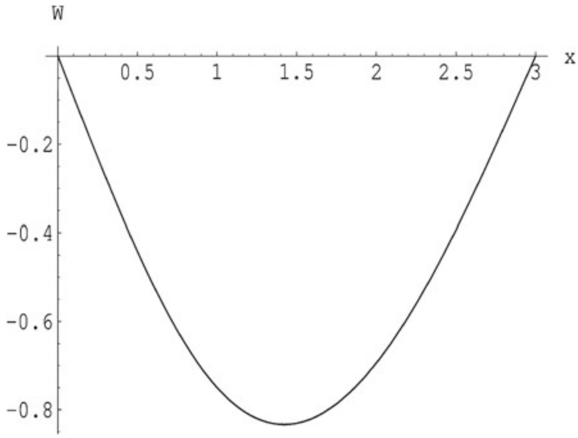


Fig. 3.14 The displacement of the beam $y(x)$ with $W = 1, k = 1$ and $P = 1$. The length l equals 3

3.5 Integral Equations

An integral equation is, as the name implies, an equation in which the unknown occurs in the form of an integral. The Laplace transform proves very useful in solving many types of integral equation, especially when the integral takes the form of a convolution. Here are some typical integral equations:

$$\int_a^b K(x, y)\phi(y)dy = f(x)$$

where K and f are known. A more general equation is:

$$A(x)\phi(x) - \lambda \int_a^b K(x, y)\phi(y)dy = f(x)$$

and an even more general equation would be:

$$\int_a^b K(x, y, \phi(y))dy = f(x).$$

In these equations, K is called the kernel of the integral equation. The general theory of how to solve integral equations is outside the scope of this text, and we shall content ourselves with solving a few special types particularly suited to solution using Laplace transforms. The last very general integral equation is non-linear and is in general very difficult to solve. The second type is called a Fredholm integral equation of the third kind. If $A(x) = 1$, this equation becomes

$$\phi(x) - \lambda \int_a^b K(x, y)\phi(y)dy = f(x)$$

which is the Fredholm integral equation of the second kind. In integral equations, x is the independent variable, so a and b can depend on it. The Fredholm integral equation covers the case where a and b are constant, in the cases where a or b (or both) depend on x the integral equation is called a Volterra integral equation. One particular case, $b = x$, is particularly amenable to solution using Laplace transforms. The following example illustrates this.

Example 3.22 *Solve the integral equation*

$$\phi(x) - \lambda \int_0^x e^{x-y}\phi(y)dy = f(x)$$

where $f(x)$ is a general function of x.

Solution The integral is in the form of a convolution; it is in fact $\lambda e^x * \phi(x)$ where * denotes the convolution operation. The integral can thus be written

$$\phi(x) - \lambda e^x * \phi(x) = f(x).$$

Taking the Laplace transform of this equation and utilising the convolution theorem gives

$$\bar{\phi} - \lambda \frac{\bar{\phi}}{s-1} = \bar{f}$$

where $\bar{\phi} = \mathcal{L}\phi$ and $\bar{f} = \mathcal{L}f$. Solving for $\bar{\phi}$ gives

$$\bar{\phi}\left(\frac{s-1-\lambda}{s-1}\right) = \bar{f}$$

$$\bar{\phi} = \frac{s-1}{s-(1+\lambda)}\bar{f} = \bar{f} + \frac{\lambda \bar{f}}{s-(1+\lambda)}.$$

So inverting gives

$$\phi = f + \lambda f * e^{(1+\lambda)x}$$

and

$$\phi(x) = f(x) + \lambda \int_0^x f(y)e^{(1+\lambda)(x-y)}dy.$$

This is the solution of the integral equation.

The solution of integral equations of these types usually involves advanced methods including complex variable techniques. These can only be understood after the methods of Chap. 8 have been introduced and are the subject of more advanced texts (e.g. Hochstadt (1989)).

3.6 Exercises

1. Given suitably well behaved functions f, g and h establish the following properties of the convolution $f * g$ where

$$f * g = \int_0^t f(\tau)g(t-\tau)d\tau.$$

(a) $f * g = g * f$, (b) $f * (g * h) = (f * g) * h$,
(c) determine f^{-1} such that $f * f^{-1} = 1$, stating any extra properties f must possess in order for the inverse f^{-1} to exist.

2. Use the convolution theorem to establish

$$\mathcal{L}^{-1}\left\{\frac{\bar{f}(s)}{s}\right\} = \int_0^t f(\tau)d\tau.$$

3. Find the following convolutions

(a) $t * \cos t$, (b) $t * t$, (c) $\sin t * \sin t$, (d) $e^t * t$, (e) $e^t * \cos t$.

4. (a) Show that

$$\lim_{x\to 0}\left(\frac{\mathrm{erf}(x)}{x}\right) = \frac{2}{\sqrt{\pi}}.$$

(b) Show that

$$\mathcal{L}\{t^{-1/2}\mathrm{erf}\sqrt{t}\} = \frac{2}{\sqrt{\pi s}}\tan^{-1}\left(\frac{1}{\sqrt{s}}\right).$$

5. Solve the following differential equations by using Laplace transforms:
(a)

$$\frac{dx}{dt} + 3x = e^{2t}, \quad x(0) = 1,$$

(b)

$$\frac{dx}{dt} + 3x = \sin t, \quad x(\pi) = 1,$$

(c)

$$\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 5x = 8\sin t, \quad x(0) = x'(0) = 0,$$

(d)

$$\frac{d^2x}{dt^2} - 3\frac{dx}{dt} - 2x = 6, \quad x(0) = x'(0) = 1,$$

(e)

$$\frac{d^2y}{dt^2} + y = 3\sin(2t), \quad y(0) = 3, \quad y'(0) = 1.$$

6. Solve the following pairs of simultaneous differential equations by using Laplace transforms:

(a)

$$\frac{dx}{dt} - 2x - \frac{dy}{dt} - y = 6e^{3t}$$
$$2\frac{dx}{dt} - 3x + \frac{dy}{dt} - 3y = 6e^{3t}$$

subject to $x(0) = 3$, and $y(0) = 0$.

(b)

$$4\frac{dx}{dt} + 6x + y = 2\sin(2t)$$
$$\frac{d^2x}{dt^2} + x - \frac{dy}{dt} = 3e^{-2t}$$

subject to $x(0) = 2$, $x'(0) = -2$, (eliminate y).

(c)

$$\frac{d^2x}{dt^2} - x + 5\frac{dy}{dt} = t$$
$$\frac{d^2y}{dt^2} - 4y - 2\frac{dx}{dt} = -2$$

subject to $x(0) = 0$, $x'(0) = 0$, $y(0) = 1$ and $y'(0) = 0$.

7. Demonstrate the phenomenon of *resonance* by solving the equation:

$$\frac{d^2x}{dt^2} + k^2x = A\sin t$$

subject to $x(0) = x_0, \quad x'(0) = v_0$ then showing that the solution is unbounded as $t \to \infty$ for particular values of k.

8. Assuming that the air resistance is proportional to speed, the motion of a particle in air is governed by the equation:

$$\frac{w}{g}\frac{d^2x}{dt^2} + b\frac{dx}{dt} = w.$$

If at $t = 0, x = 0$ and

$$v = \frac{dx}{dt} = v_0$$

show that the solution is

$$x = \frac{gt}{a} + \frac{(av_0 - g)(1 - e^{-at})}{a^2}$$

where $a = bg/w$. Deduce the terminal speed of the particle.

9. Determine the algebraic system of equations obeyed by the transformed electrical variables $\bar{j_1}, \bar{j_2}, \bar{j_3}$ and $\bar{q_3}$ given the electrical circuit equations

$$\begin{aligned} j_1 &= j_2 + j_3 \\ R_1 j_1 + L_2\frac{dj_2}{dt} &= E\sin(\omega t) \\ R_1 j_1 + R_3 j_3 + \frac{1}{C}q_3 &= E\sin(\omega t) \\ j_3 &= \frac{dq_3}{dt} \end{aligned}$$

where, as usual, the overbar denotes the Laplace transform.

10. Solve the loaded beam problem expressed by the equation

$$k\frac{d^4y}{dx^4} = \frac{w_0}{c}[c - x + (x - c)H(x - c)] \quad \text{for } 0 < x < 2c$$

subject to the boundary conditions $y(0) = 0$, $y'(0) = 0$, $y''(2c) = 0$ and $y'''(2c) = 0$. $H(x)$ is the Heaviside unit step function. Find the bending moment ky'' at the point $x = \frac{1}{2}c$.

11. Solve the integral equation

$$\phi(t) = t^2 + \int_0^t \phi(y) \sin(t - y) dy.$$

Chapter 4
Fourier Series

4.1 Introduction

Before getting to Fourier series proper, we need to discuss the context. To understand why Fourier series are so useful, one uses the properties of an inner product space and that trigonometric functions are an example of one. It is the properties of the inner product space, coupled with the analytically familiar properties of the sine and cosine functions that give Fourier series their usefulness and power.

The basic assumption behind Fourier series is that any given function can be expressed in terms of a series of sine and cosine functions, and that once found the series is unique. Stated coldly with no preliminaries this sounds preposterous, but to those familiar with the theory of linear spaces it is not. All that is required is that the sine and cosine functions are a *basis* for the linear space of functions to which the given function belongs. Some details are given in Appendix C. Those who have a background knowledge of linear algebra sufficient to absorb this appendix should be able to understand the following two theorems which are essential to Fourier series. They are given without proof and may be ignored by those willing to accept the results that depend on them. The first result is Bessel's inequality. It is conveniently stated as a theorem.

Theorem 4.1 **(Bessel's Inequality)** *If*

$$\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n, \ldots\}$$

is an orthonormal basis for the linear space V, then for each $\mathbf{a} \in V$ the series

$$\sum_{r=1}^{\infty} |\langle \mathbf{a}, \mathbf{e}_n \rangle|^2$$

converges. In addition, the inequality

P. Dyke, *An Introduction to Laplace Transforms and Fourier Series*,
Springer Undergraduate Mathematics Series, DOI: 10.1007/978-1-4471-6395-4_4,

$$\sum_{r=1}^{\infty} |\langle \mathbf{a}, \mathbf{e}_n \rangle|^2 \leq ||\mathbf{a}||^2$$

holds.

An important consequence of Bessel's inequality is the Riemann–Lebesgue lemma. This is also stated as a theorem:-

Theorem 4.2 (Riemann–Lebesgue) *Let* $\{\mathbf{e}_1, \mathbf{e}_2, \ldots\}$ *be an orthonormal basis of infinite dimension for the inner product space* V. *Then, for any* $\mathbf{a} \in V$

$$\lim_{n \to \infty} \langle \mathbf{a}, \mathbf{e}_n \rangle = 0.$$

This theorem in fact follows directly from Bessel's inequality as the nth term of the series on the right of Bessel's inequality must tend to zero as n tends to ∞.

Although some familiarity with analysis is certainly a prerequisite here, there is merit in emphasising the two concepts of pointwise convergence and uniform convergence. It will be out of place to go into proofs, but the difference is particularly important to the study of Fourier series as we shall see later. Here are the two definitions.

Definition 4.1 (Pointwise Convergence) *Let*

$$\{f_0, f_1, \ldots, f_m, \ldots\}$$

be a sequence of functions defined on the closed interval $[a, b]$. *We say that the sequence* $\{f_0, f_1, \ldots, f_m, \ldots\}$ *converges pointwise to* f *on* $[a, b]$ *if for each* $x \in [a, b]$ *and* $\epsilon > 0$ *there exists a natural number* $N(\epsilon, x)$ *such that*

$$|f_m(x) - f(x)| < \epsilon$$

for all $m \geq N(\epsilon, x)$.

Definition 4.2 (Uniform Convergence) *Let*

$$\{f_0, f_1, \ldots, f_m, \ldots\}$$

be a sequence of functions defined on the closed interval $[a, b]$. *We say that the sequence* $\{f_0, f_1, \ldots, f_m, \ldots\}$ *converges uniformly to* f *on* $[a, b]$ *if for each* $\epsilon > 0$ *there exists a natural number* $N(\epsilon)$ *such that*

$$|f_m(x) - f(x)| < \epsilon$$

for all $m \geq N(\epsilon)$ *and for all* $x \in [a, b]$.

It is the difference and not the similarity of these two definitions that is important. All uniformly convergent sequences are pointwise convergent, but not vice versa. This is because N in the definition of pointwise convergence depends on x; in the definition uniform convergence it does not which makes uniform convergence a global rather than a local property. The N in the definition of uniform convergence will do for any x in $[a, b]$.

Armed with these definitions and assuming a familiarity with linear spaces, we will eventually go ahead and find the Fourier series for a few well known functions. We need a few more preliminaries before we can do this.

4.2 Definition of a Fourier Series

As we have said, Fourier series consist of a series of sine and cosine functions. We have also emphasised that the theory of linear spaces can be used to show that it possible to represent any periodic function to any desired degree of accuracy provided the function is periodic and piecewise continuous. To start, it is easiest to focus on functions that are defined in the closed interval $[-\pi, \pi]$. These functions will be piecewise continuous and they will possess one sided limits at $-\pi$ and π. So, using mathematical notation, we have $f : [-\pi, \pi] \to \mathbf{C}$. The restriction to this interval will be lifted later, but periodicity will always be essential.

It also turns out that the points at which f is discontinuous need not be points at which f is defined uniquely. As an example of what is meant, Fig. 4.1 shows three possible values of the function

$$f_a = \begin{cases} 0 & t < 1 \\ 1 & t > 1 \end{cases}$$

at $t = 1$. These are $f_a(1) = 0$, $f_a(1) = 1$ and $f_a(1) = 1/2$, and, although we do need to be consistent in order to satisfy the need for $f_a(t)$ to be well defined, in theory it does not matter exactly where $f_a(1)$ is. However, Fig. 4.1c is the right choice for Fourier series; the following theorem due to Dirichlet tells us why.

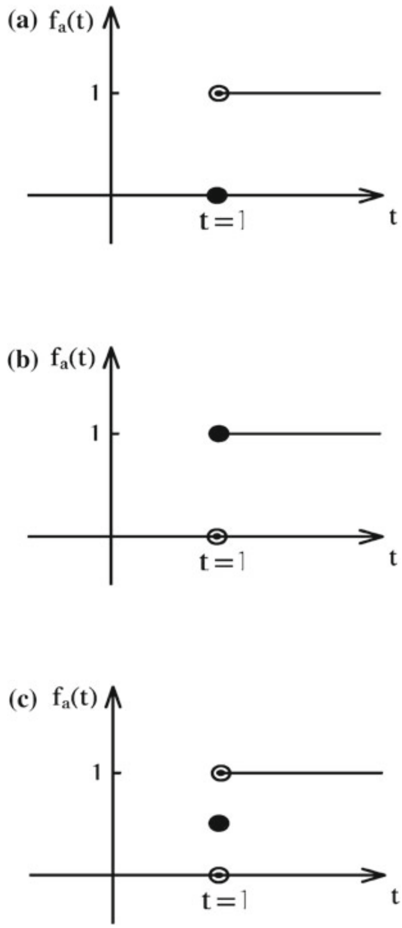
Theorem 4.3 *If f is a member of the space of piecewise continuous functions which are 2π periodic on the closed interval $[-\pi, \pi]$ and which has both left and right derivatives at each $x \in [-\pi, \pi]$, then for each $x \in [-\pi, \pi]$ the Fourier series of f converges to the value*

$$\frac{f(x_-) + f(x_+)}{2}.$$

At both end points, $x = \pm\pi$, the series converges to

$$\frac{f(\pi_-) + f((-\pi)_+)}{2}.$$

Fig. 4.1 **a** $f_a(1) = 0$, **b** $f_a(1) = 1$, **c** $f_a(1) = 1/2$



The proof of this is beyond the scope of this book, but some comments are usefully made. If x is a point at which the function f is continuous, then

$$\frac{f(x_-) + f(x_+)}{2} = f(x)$$

and the theorem is certainly eminently plausible as any right hand side other than $f(x)$ for this mean of left and right sided limits would be preposterous. It is still however difficult to prove rigorously. At other points, including the end points, the theorem gives the useful result that at points of discontinuity the value of the Fourier series for f takes the mean of the one sided limits of f itself at the discontinuous point. Given that the Fourier series is a continuous function (assuming the series to be uniformly convergent) representing f at this point of discontinuity this is the best that we can expect. Dirichlet's theorem is not therefore surprising. The formal proof of the theorem can be found in graduate texts such as Pinkus and Zafrany (1997) and depends on careful application of the Riemann–Lebesgue lemma and Bessel's inequality. Since f is periodic of period 2π, $f(\pi) = f(-\pi)$ and the last part of the theorem is seen to be nothing special, merely a re-statement that the Fourier series takes the mean of the one sided limits of f at discontinuous points.

We now state the basic theorem that enables piecewise continuous functions to be able to be expressed as Fourier series. The linear space notation is that used earlier (see Appendix C) to which you are referred for more details.

Theorem 4.4 *The sequence of functions*

$$\left\{\frac{1}{\sqrt{2}}, \sin(x), \cos(x), \sin(2x), \cos(2x), \ldots\right\}$$

form an infinite orthonormal sequence in the space of all piecewise continuous functions on the interval $[-\pi, \pi]$ *where the inner product* $\langle f, g\rangle$ *is defined by*

$$\langle f, g\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f\bar{g}dx$$

the overbar denoting complex conjugate.

Proof First we have to establish that $\langle f, g\rangle$ is indeed an inner product over the space of all piecewise continuous functions on the interval $[-\pi, \pi]$. The integral

$$\int_{-\pi}^{\pi} f\bar{g}dx$$

certainly exists. As f and $\bar{g}$ are piecewise continuous, so is the product $f\bar{g}$ and hence it is (Riemann) integrable. From elementary properties of integration it is easy to deduce that the space of all piecewise continuous functions is indeed an inner product space. There are no surprises. 0 and 1 are the additive and multiplicative identities, $-f$ is the additive inverse and the rules of algebra ensure associativity, distributivity and commutativity. We do, however spend some time establishing that the set

$$\left\{\frac{1}{\sqrt{2}}, \sin(x), \cos(x), \sin(2x), \cos(2x), \ldots\right\}$$

is orthonormal. To do this, it will be sufficient to show that

$$\left\langle\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right\rangle = 1, \quad \langle\sin(nx), \sin(nx)\rangle = 1,$$

$$\langle\cos(nx), \cos(nx)\rangle = 1, \quad \left\langle\frac{1}{\sqrt{2}}, \sin(nx)\right\rangle = 0,$$

$$\left\langle \frac{1}{\sqrt{2}}, \cos(nx) \right\rangle = 0, \quad \langle \cos(mx), \sin(nx) \rangle = 0,$$

$$\langle \cos(mx), \sin(nx) \rangle = 0, \quad \langle \sin(mx), \sin(nx) \rangle = 0,$$

with $m \neq n$; $m, n = 1, 2, \ldots$. Time spent on this is time well spent as orthonormality lies behind most of the important properties of Fourier series. For this, we do not use short cuts.

$$\begin{aligned}
\left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle &= \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{1}{2} dx = 1 \text{ trivially} \\
\langle \sin(nx), \sin(nx) \rangle &= \frac{1}{\pi} \int_{-\pi}^{\pi} \sin^2(nx) dx \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} (1 - \cos(2nx)) dx = 1 \text{ for all } n \\
\langle \cos(nx), \cos(nx) \rangle &= \frac{1}{\pi} \int_{-\pi}^{\pi} \cos^2(nx) dx \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} (1 + \cos(2nx)) dx = 1 \text{ for all } n \\
\left\langle \frac{1}{\sqrt{2}}, \cos(nx) \right\rangle &= \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{1}{\sqrt{2}} \cos(nx) dx \\
&= \frac{1}{\pi\sqrt{2}} \left[\frac{1}{n} \sin(nx) \right]_{-\pi}^{\pi} = 0 \text{ for all } n \\
\left\langle \frac{1}{\sqrt{2}}, \sin(nx) \right\rangle &= \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{1}{\sqrt{2}} \sin(nx) dx \\
&= \frac{1}{\pi\sqrt{2}} \left[-\frac{1}{n} \cos(nx) \right]_{-\pi}^{\pi} \\
&= \frac{1}{n\pi\sqrt{2}} \left((-1)^n - (-1)^n \right) = 0 \text{ for all } n \\
\langle \cos(mx), \sin(nx) \rangle &= \frac{1}{\pi} \int_{-\pi}^{\pi} \cos(mx) \sin(nx) dx \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\sin((m+n)x) + \sin((m-n)x) \right) dx \\
&= \frac{1}{2\pi} \left[-\frac{\cos((m+n)x)}{m+n} - \frac{\cos((m-n)x)}{m-n} \right]_{-\pi}^{\pi} = 0, \ (m \neq n)
\end{aligned}$$

since the function in the square bracket is the same at both $-\pi$ and π. If $m = n$, $\sin((m-n)x) = 0$ but otherwise the arguments go through unchanged and $\langle \cos(mx), \sin(mx) \rangle = 0$, $m, n = 1, 2 \ldots$. Now

$$\begin{aligned}
\langle \cos(mx), \cos(nx) \rangle &= \frac{1}{\pi} \int_{-\pi}^{\pi} \cos(mx)\cos(nx)dx \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} (\cos((m+n)x) + \cos((m-n)x))\, dx \\
&= \frac{1}{2\pi} \left[\frac{\sin((m+n)x)}{m+n} + \frac{\sin((m-n)x)}{m-n} \right]_{-\pi}^{\pi} \quad \text{as } m \neq n \\
&= 0 \text{ as all functions are zero at both limits.}
\end{aligned}$$

Finally,

$$\begin{aligned}
\langle \sin(mx), \sin(nx) \rangle &= \frac{1}{\pi} \int_{-\pi}^{\pi} \sin(mx)\sin(nx)dx \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} (\cos((m-n)x) - \cos((m+n)x))\, dx \\
&= 0 \text{ similarly to the previous result.}
\end{aligned}$$

Hence the theorem is firmly established. □

We have in the above theorem shown that the sequence

$$\left\{ \frac{1}{\sqrt{2}}, \sin(x), \cos(x), \sin(2x), \cos(2x), \ldots \right\}$$

is orthogonal. It is in fact also true that this sequence forms a basis (an *orthonormal* basis) for the space of piecewise continuous functions in the interval $[-\pi, \pi]$. This and other aspects of the theory of linear spaces, an outline of which is given in Appendix C. All this thus ensures that an arbitrary element of the linear space of piecewise continuous functions can be expressed as a linear combination of the elements of this sequence, i.e.

$$\begin{aligned}
f(x) \sim \frac{a_0}{\sqrt{2}} &+ a_1 \cos(x) + a_2 \cos(2x) + \cdots + a_n \cos(nx) + \cdots \\
&+ b_1 \sin(x) + b_2 \sin(2x) + \cdots + b_n \sin(nx) + \cdots
\end{aligned}$$

so

$$f(x) \sim \frac{a_0}{\sqrt{2}} + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx)) \qquad -\pi < x < \pi \tag{4.1}$$

with the tilde being interpreted as follows. At points of discontinuity, the left hand side is the mean of the two one sided limits as dictated by Dirichlet's theorem. At points where the function is continuous, the right-hand side converges to $f(x)$ and the tilde means equals. This is the "standard" Fourier series expansion for $f(x)$ in the range $-\pi < x < \pi$. Since the right hand side is composed entirely of periodic functions, period 2π, it is necessary that

$$f(x) = f(x + 2N\pi) \qquad N = 0, \pm 1, \pm 2, \ldots.$$

The authors of engineering texts are happy to start with Eq. 4.1, then by multiplying through by $\sin(nx)$ (say) and integrating term by term between $-\pi$ and π, all but the b_n on the right disappears. This gives

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx.$$

Similarly

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx.$$

Of course, this gives the correct results, but questions about the legality or otherwise of dealing with and manipulating infinite series remain. In the context of linear spaces we can immediately write

$$f(x) = (a_0, a_1, b_1, \ldots, a_n, b_n, \ldots)$$

is a vector expressed in terms of the orthonormal basis

$$e = (e_0, e_{\alpha_1}, e_{\beta_1}, \ldots, e_{\alpha_n}, e_{\beta_n}, \ldots)$$

$$e_0 = \frac{1}{\sqrt{2}}, e_{\alpha_n} = \cos(nx), \quad e_{\beta_n} = \sin(nx),$$

so

$$\langle f, e_{\alpha_n}\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx$$

and

$$\langle f, e_{\beta_n}\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx.$$

The series

$$f = \sum_{k=0}^{\infty}\langle f, e_k\rangle e_k$$

where $e_k,\ k = 0, \ldots\ldots$ is a renumbering of the basis vectors. This is the standard expansion of f in terms of the orthonormal basis and is the Fourier series for f. Invoking the linear space theory therefore helps us understand how it is possible to express any function piecewise continuous in $[-\pi, \pi]$ as the series expansion (4.1),

$$f(x) \sim \frac{a_0}{\sqrt{2}} + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx)) \quad -\pi < x < \pi$$

where

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) dx,$$

and

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) dx,\ n = 0, 1, 2, \ldots.$$

and remembering to interpret correctly the left-hand side at discontinuities to be in line with Dirichlet's theorem. It is also now clear why a_0 is multiplied by $1/\sqrt{2}$ for together with e_0 it enables a uniform definition of a_n, but it is not convenient and using $\frac{1}{\sqrt{2}}a_0$ as the constant term is almost never done in practise. Students need to remember this when doing calculations. It is a common source of error. Books differ as to where the factor goes. Some use a_0 with the factor $1/2$ in a separate integral for a_0. We use $\frac{1}{2}a_0$ as the constant term (so $e_0 = 1$ now), and the practical Fourier series is:

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx)) \quad -\pi < x < \pi$$

where

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) dx,$$

and

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) dx,\ n = 0, 1, 2, \ldots.$$

We are now ready to do some practical examples. There is good news for those who perhaps are a little impatient with all this theory. It is not at all necessary to understand about linear space theory in order to calculate Fourier series. The earlier theory gives the framework in which Fourier series operate as well as enabling us to give decisive answers to key questions that can arise in awkward or controversial cases, for example if the existence or uniqueness of a particular Fourier series is in question. The first example is not controversial.

Example 4.1 *Determine the Fourier series for the function*

$$
\begin{aligned}
f(x) &= 2x + 1 \qquad -\pi < x < \pi \\
f(x) &= f(x + 2\pi) \quad x \in \mathbf{R}
\end{aligned}
$$

where Theorem 4.3 applies at the end points.

Solution As $f(x)$ is obviously piecewise continuous in $[-\pi, \pi]$, in fact the only discontinuities occurring at the end points, we simply use the formulae

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx,$$

and

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx$$

to determine the Fourier coefficients. Now,

$$
\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi}(2x+1)\cos(nx)dx \\
&= \frac{1}{\pi}\left[\frac{(2x+1)}{n}\sin(nx)\right]_{-\pi}^{\pi} - \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{2}{n}\sin(nx)dx \quad (n \neq 0) \\
&= 0 + \frac{1}{\pi}\frac{2}{n^2}\left[\cos(nx)\right]_{-\pi}^{\pi} = 0
\end{aligned}
$$

since $\cos(n\pi) = (-1)^n$. If $n = 0$ then

$$
\begin{aligned}
a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi}(2x+1)dx \\
&= \frac{1}{\pi}\left[x^2 + x\right]_{-\pi}^{\pi} = 2 \\
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi}(2x+1)\sin(nx)dx \\
&= \frac{1}{\pi}\left[-\frac{(2x+1)}{n}\cos(nx)\right]_{-\pi}^{\pi} + \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{2}{n}\cos(nx)dx \\
&= \frac{1}{\pi}\left[-\frac{(2\pi+1)}{n} + \frac{(-2\pi+1)}{n}\right](-1)^n \\
&= -\frac{4}{n}(-1)^n.
\end{aligned}
$$

Hence

$$f(x) \sim 1 - 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin(nx), \quad -\pi < x < \pi.$$

From this series, we can deduce the Fourier series for x as follows:-

$$\text{we have } 2x + 1 \sim 1 - 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin(nx), \quad x \in [-\pi, \pi]$$

so

$$x \sim 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(nx)$$

gives the Fourier series for x in $[-\pi, \pi]$.

Thus the Fourier series for the general straight line $y = mx + c$ in $[-\pi, \pi]$ must be

$$y \sim c - 2m\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin(nx).$$

The explanation for the lack of cosine terms in these Fourier series follows later after the discussion of even and odd functions. Here is a slightly more involved example.

Example 4.2 *Find the Fourier series for the function*

$$f(x) = e^x, \quad -\pi < x < \pi$$

$$f(x + 2\pi) = f(x), \quad x \in \mathbf{R}$$

where Theorem 4.3 applies at the end points.

Solution This problem is best tackled by using the power of complex numbers. We start with the two standard formulae:

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x \cos(nx)dx$$

and

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x \sin(nx)dx$$

and form the sum $a_n + ib_n$ where $i = \sqrt{-1}$. The integration is then quite straightforward

$$\begin{aligned}
a_n + ib_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} e^{x+inx}\,dx \\
&= \frac{1}{\pi(1+in)}[e^{x+inx}]_{-\pi}^{\pi} \\
&= \frac{1}{\pi(1+in)}[e^{(1+in)\pi} - e^{-(1+in)\pi}] \\
&= \frac{(-1)^n}{\pi(1+in)}(e^{\pi} - e^{-\pi}) \text{ since } e^{in\pi} = (-1)^n \\
&= \frac{2(-1)^n \sinh\pi}{\pi(1+n^2)}(1-in).
\end{aligned}$$

Hence, taking real and imaginary parts we obtain

$$a_n = (-1)^n \frac{2\sinh(\pi)}{\pi(1+n^2)}, \quad b_n = -\frac{2n(-1)^n \sinh(\pi)}{\pi(1+n^2)}.$$

In this example a_0 is given by

$$a_0 = \frac{2}{\pi}\sinh(\pi),$$

hence giving the Fourier series as

$$f(x) = \frac{\sinh(\pi)}{\pi} + \frac{2}{\pi}\sinh(\pi)\sum_{n=1}^{\infty}\frac{(-1)^n}{1+n^2}(\cos(nx) - n\sin(nx)), \quad -\pi < x < \pi.$$

Let us take this opportunity to make use of this series to find the values of some infinite series. The above series is certainly valid for $x = 0$ so inserting this value into both sides of the above equation and noting that $f(0)(= e^0) = 1$ gives

$$1 = \frac{\sinh(\pi)}{\pi} + \frac{2}{\pi}\sinh(\pi)\sum_{n=1}^{\infty}\frac{(-1)^n}{1+n^2}.$$

Thus

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{1+n^2} = \frac{\pi}{2}\operatorname{cosech}(\pi) - \frac{1}{2}$$

and

$$\sum_{n=-\infty}^{\infty}\frac{(-1)^n}{1+n^2} = 2\sum_{n=1}^{\infty}\frac{(-1)^n}{1+n^2} + 1 = \pi\operatorname{cosech}(\pi).$$

Further, there is the opportunity here to use Dirichlet's theorem. Putting $x = \pi$ into the Fourier series for e^x is not strictly legal. However, Dirichlet's theorem states that the value of the series at this discontinuity is the mean of the one sided limits either side which is $\frac{1}{2}(e^{-\pi} + e^{\pi}) = \cosh \pi$. The Fourier series evaluated at $x = \pi$ is

$$\frac{\sinh(\pi)}{\pi} + \frac{2}{\pi}\sinh(\pi)\sum_{n=1}^{\infty}\frac{1}{1+n^2}$$

as $\cos(n\pi) = (-1)^n$ and $\sin(n\pi) = 0$ for all integers n. The value of $f(x)$ at $x = \pi$ is taken to be that dictated by Dirichlet's theorem, viz. $\cosh(\pi)$. We therefore equate these expressions and deduce the series

$$\sum_{n=1}^{\infty}\frac{1}{1+n^2} = \frac{\pi}{2}\coth(\pi) - \frac{1}{2}$$

and

$$\sum_{n=-\infty}^{\infty}\frac{1}{1+n^2} = 2\sum_{n=1}^{\infty}\frac{1}{1+n^2} + 1 = \pi\coth(\pi).$$

Having seen the general method of finding Fourier series, we are ready to remove the restriction that all functions have to be of period 2π and defined in the range $[-\pi, \pi]$. The most straightforward way of generalising to Fourier series of any period is to effect the transformation $x \to 2\pi x/l$ where l is assigned by us. Thus if $x \in [-\pi, \pi]$, $\pi x/l \in [-l, l]$. Since $\cos(\pi x/l)$ and $\sin(\pi x/l)$ have period $2l$ the Fourier series valid in $[-l, l]$ takes the form

$$f(x) \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n\cos\left(\frac{n\pi x}{l}\right) + b_n\sin\left(\frac{n\pi x}{l}\right)\right), \quad -l < x < l,$$

where

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx \tag{4.2}$$

and

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx. \tag{4.3}$$

The examples of finding this kind of Fourier series are not remarkable, they just contain (in general) messier algebra. Here is just one example.

Example 4.3 *Determine the Fourier Series of the function*

$$f(x) = |x|, \quad -3 \le x \le 3,$$
$$f(x) = f(x+6).$$

Solution The function $f(x) = |x|$ is continuous, therefore we can use Eqs. 4.2 and 4.3 to generate the Fourier coefficients. First of all

$$a_0 = \frac{1}{3}\int_{-3}^{3} |x| dx = 3.$$

Secondly,

$$\begin{aligned}
a_n &= \frac{1}{3}\int_{-3}^{3} |x| \cos\left(\frac{n\pi x}{3}\right) dx \\
&= \frac{1}{3}\int_{0}^{3} x \cos\left(\frac{n\pi x}{3}\right) dx \\
&= \frac{2}{3}\left[\frac{3x}{n\pi} \sin\left(\frac{n\pi x}{3}\right)\right]_0^3 - \frac{2}{3}\int_0^3 \frac{3}{n\pi} \sin\left(\frac{n\pi x}{3}\right) dx \\
&= 0 + \frac{2}{n\pi}\left[\frac{3}{n\pi} \cos\left(\frac{n\pi x}{3}\right)\right]_0^3 \\
\text{so } a_n &= \frac{6}{(n\pi)^2}[-1 + (-1)^n].
\end{aligned}$$

This is zero if n is even and $-12/(n\pi)^2$ if n is odd. Hence

$$a_{2k+1} = -\frac{12}{(2k+1)^2\pi^2}, \quad k = 0, 1, 2, \ldots$$

Similarly,

$$\begin{aligned}
b_n &= \frac{1}{3}\int_{-3}^{3} |x| \sin\left(\frac{n\pi x}{3}\right) dx \\
&= -\frac{1}{3}\int_{-3}^{0} x \sin\left(\frac{n\pi x}{3}\right) dx + \frac{1}{3}\int_{0}^{3} x \sin\left(\frac{n\pi x}{3}\right) dx \\
&= 0 \text{ for all } n.
\end{aligned}$$

Hence

$$f(x) = \frac{3}{2} - \frac{12}{\pi^2}\sum_{n=0}^{\infty} \frac{1}{(2n-1)^2} \cos\left[\frac{2n-1}{3}\pi x\right], \quad 0 \le x \le 3,$$

note the equality; $f(x)$ is continuous on $[0, 3]$ hence there is no discontinuity at the end points and no need to invoke Dirichlet's theorem.

4.3 Odd and Even Functions

This topic is often met at a very elementary level at schools in the context of how graphs of functions look on the page in terms of symmetries. However, here we give formal definitions and, more importantly, see how the identification of oddness or evenness in functions literally halves the amount of work required in finding the Fourier series.

Definition 4.3 *A function $f(x)$ is termed even with respect to the value a if*

$$f(a + x) = f(a - x)$$

for all values of x.

Definition 4.4 *A function $f(x)$ is termed odd with respect to the value a if*

$$f(a + x) = -f(a - x)$$

for all values of x.

The usual expressions "$f(x)$ is an even function" and "$f(x)$ is an odd function" means that $a = 0$ has been assumed. i.e. $f(x) = f(-x)$ means f is an even function and $f(x) = -f(-x)$ means f is an odd function. Well known even functions are:-

$$|x|, x^2, \cos(x).$$

Well known odd functions are

$$x, \sin(x), \tan(x).$$

An even function of x, plotted on the (x, y) plane, is symmetrical about the y axis. An odd function of x drawn on the same axes is anti-symmetric (see Fig. 4.2).

The important consequence of the essential properties of these functions is that the Fourier series of an even function has to consist entirely of even functions and therefore has no sine terms. Similarly, the Fourier series of an odd function must consist entirely of odd functions, i.e. only sine terms.

Hence, given

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx)) \quad -\pi < x < \pi$$

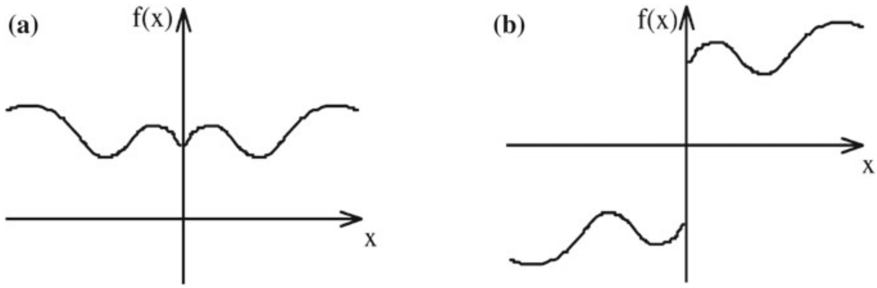


Fig. 4.2 **a** An even function, **b** an odd function

if $f(x)$ is even for all x then $b_n = 0$ for all n. If $f(x)$ is odd for all x then $a_n = 0$ for all n. We have already had one example of this. The function x is odd, and the Fourier series found after Example 4.1 is

$$x = 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(nx), \quad -\pi < x < \pi$$

which has no even terms.

Example 4.4 *Determine the Fourier series for the function*

$$\begin{aligned} f(x) &= x^2 \\ f(x) &= f(x+2\pi), \quad -\pi \le x \le \pi. \end{aligned}$$

Solution Since $x^2 = (-x)^2$, $f(x)$ is an even function. Thus the Fourier series consists solely of even functions which means $b_n = 0$ for all n. We therefore compute the a_n's as follows

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 dx \\ &= \frac{1}{3}\frac{1}{\pi}[x^3]_{-\pi}^{\pi} = \frac{2}{3}\pi^2 \\ \text{also } a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cos(nx)dx \quad (n \neq 0) \\ &= \frac{1}{\pi}\left[\frac{x^2}{n}\sin(nx)\right]_{-\pi}^{\pi} - \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{2x}{n}\sin(nx)dx \\ &= 0 + \frac{1}{\pi}\left[\frac{2x}{n^2}\cos(nx)\right]_{-\pi}^{\pi} - \frac{2}{\pi n}\int_{-\pi}^{\pi} \frac{\cos(nx)}{n}dx \\ &= \frac{4}{n^2}(-1)^n \end{aligned}$$

so

$$f(x) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx \quad -\pi \le x < \pi.$$

This last example leads to some further insights.

If we let $x = 0$ in the Fourier series just obtained, the right-hand side is

$$\frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

hence

$$0 = \frac{\pi^2}{3} + 4\left(-1 + \frac{1}{2^2} - \frac{1}{3^2} + \cdots\right)$$

so

$$1 - \frac{1}{2^2} + \frac{1}{3^2} - \cdots = \frac{\pi^2}{12}.$$

This is interesting in itself as this series is not easy to sum. However it is also possible to put $x = \pi$ and obtain

$$\pi^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{1}{n^2}$$

i.e.

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Note that even periodic functions are continuous, whereas odd periodic functions are, using Dirichlet's theorem, zero at the end points. We shall utilise the properties of odd and even functions from time to time usually in order to simplify matters and reduce the algebra. Another tool that helps in this respect is the complex form of the Fourier series which is derived next.

4.4 Complex Fourier Series

Given that a Fourier series has the general form

$$f(x) \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx)), \quad -\pi < x < \pi, \tag{4.4}$$

we can write

$$\cos(nx) = \frac{1}{2}(e^{inx} + e^{-inx})$$

and

$$\sin(nx) = \frac{1}{2i}(e^{inx} - e^{-inx}).$$

If these equations are inserted into Eq. 4.4 then we obtain

$$f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

where

$$c_n = \frac{1}{2}(a_n - ib_n), \quad c_{-n} = \frac{1}{2}(a_n + ib_n), \quad c_0 = \frac{1}{2}a_0, \quad n = 1, 2, \ldots.$$

Using the integrals for a_n and b_n we get

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx \text{ and } c_{-n} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{inx}dx.$$

This is called the complex form of the Fourier series and can be useful for the computation of certain types of Fourier series. More importantly perhaps, it enables the step to Fourier transforms to be made which not only unites this section and its subject, Fourier series, to the Laplace transforms that have already met, but leads naturally to applications to the field of signal processing which is of great interest to many electrical engineers.

Example 4.5 *Find the Fourier series for the function*

$$f(t) = t^2 + t, \qquad -\pi \le t \le \pi,$$
$$f(t) = f(t + 2\pi).$$

Solution We could go ahead and find the Fourier series in the usual way. However it is far easier to use the complex form but in a tailor-made way as follows. Given

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)dt$$

and

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(nt)dt$$

we define

$$d_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)e^{int}dt = a_n + ib_n$$

so that

$$\begin{aligned}
d_n &= \frac{1}{\pi}\int_{-\pi}^{\pi}(t^2+t)e^{int}dt \\
&= \frac{1}{\pi}\left[\frac{t^2+t}{in}e^{int}\right]_{-\pi}^{\pi} - \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{2t+1}{in}e^{int}dt \\
&= \frac{1}{\pi}\left[\frac{t^2+t}{in}e^{int} - \frac{2t+1}{(in)^2}e^{int} + \frac{2}{(in)^3}e^{int}\right]_{-\pi}^{\pi} \\
&= \frac{1}{\pi}\left[\frac{\pi^2+\pi}{in} - \frac{\pi^2-\pi}{in} + \frac{2\pi+1}{n^2} - \frac{-2\pi+1}{n^2}\right](-1)^n \\
&= (-1)^n\left(\frac{4}{n^2}+\frac{2}{in}\right)
\end{aligned}$$

so

$$a_n = \frac{4}{n^2}(-1)^n, \quad b_n = \frac{2}{n}(-1)^{n+1}$$

and

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi}(t^2+t)dt = \frac{2}{3}\pi^2.$$

These can be checked by direct computation. The Fourier series is thus, assuming convergence of the right hand side,

$$f(t) \sim \frac{1}{3}\pi^2 + \sum_{n=1}^{\infty}(-1)^n\left(\frac{4\cos(nt)}{n^2} - \frac{2\sin(nt)}{n}\right) \quad -\pi < t < \pi.$$

We shall use this last example to make a point that should be obvious at least with hindsight. In Example 4.1 we deduced the result

$$f(x) = 2x+1 \sim 1 - 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin(nx), \quad -\pi \le x \le \pi.$$

The right hand side is pointwise convergent for all $x \in [-\pi, \pi]$. It is therefore legal (see Sect. 4.6) to integrate the above Fourier series term by term indefinitely. The left hand side becomes

$$\int_0^t (2x+1)dx = t^2 + t.$$

The right hand side becomes

$$t - 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}\cos(nt).$$

This result may seem to contradict the Fourier series just obtained, viz.

$$t^2 + t \sim \frac{1}{3}\pi^2 + \sum_{n=1}^{\infty}(-1)^n\left(\frac{4\cos(nt)}{n^2} - \frac{2\sin(nt)}{n}\right)$$

as the right hand sides are not the same. There is no contradiction however as

$$t - 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}\cos(nt)$$

is not a Fourier series due to the presence of the isolated t at the beginning.

From a practical point of view, it is useful to know just how many terms of a Fourier series need to be calculated before a reasonable approximation to the periodic function is obtained. The answer of course depends on the specific function, but to get an idea of the approximation process, consider the function

$$f(t) = t + 2 \quad 0 \le t \le 2 \quad \text{with} \quad f(t) = f(t+2)$$

which formally has the Fourier series

$$f(t) = 3 - \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{\sin(\pi t)}{n}.$$

The sequence formed by the first seven partial sums of this Fourier series are shown superimposed in Fig. 4.3. In this instance, it can be seen that there is quite a rapid convergence to the “saw-tooth” function $f(t) = t + 2$. Problems arise where there are rapid changes of gradient (at the corners) and in trying to approximate a vertical line via trigonometric series (which brings us back to Dirichlet’s theorem). The overshoots at corners (Gibbs phenomenon) and other problems (e.g. aliasing) are treated in depth in specialist texts, but see the end of Sect. 7.6 for a bit more on this. Here we concentrate on finding the series itself and now move on to some refinements.

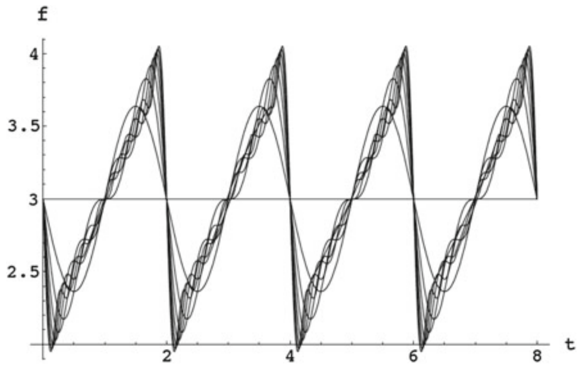


Fig. 4.3 The first seven partial sums of the Fourier series for the function $f(t) = t + 2$, $0 \le t \le 2$, $f(t) = f(t + 2)$ drawn in the range $0 \le t \le 8$

4.5 Half Range Series

There is absolutely no problem explaining half range series in terms of the normed linear space theory of Sect. 4.1. However, we shall postpone this until half range series have been explained and developed in terms of even and odd functions. This is entirely natural, at least for the applied mathematician.

Half range series are, as the name implies, series defined over half of the normal range. That is, for standard trigonometric Fourier series the function $f(x)$ is defined only in $[0, \pi]$ instead of $[-\pi, \pi]$. The value that $f(x)$ takes in the other half of the interval, $[-\pi, 0]$ is free to be defined. If we take

$$f(x) = f(-x)$$

that is $f(x)$ is even, then the Fourier series for $f(x)$ can be entirely expressed in terms of even functions, i.e cosine terms. If on the other hand

$$f(x) = -f(-x)$$

then $f(x)$ is an odd function and the Fourier series is correspondingly odd and consists only of sine terms. We are not defining the same function as two different Fourier series, for $f(x)$ is different, at least over half the range (see Fig. 4.4). We are now ready to derive the half range series in detail. First of all, let us determine the cosine series. Suppose

$$f(x) = f(x + 2\pi), \quad -\pi \le x \le \pi$$

and, additionally, $f(x) = f(-x)$ so that $f(x)$ is an even function. Since the formulae

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) dx$$

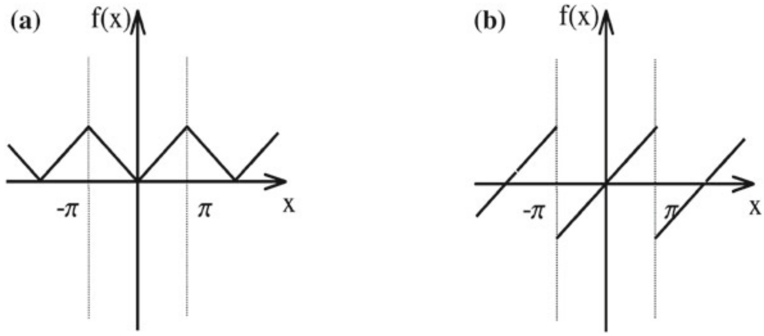


Fig. 4.4 **a** $f(x)$ is an even function (cosine series), **b** $f(x)$ is an odd function (sine series)

and

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx$$

have already been derived, we impose the condition that $f(x)$ is even. It is then easy to see that

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx dx$$

and $b_n = 0$. For odd functions, the formulae for a_n and b_n are $a_n = 0$ and

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)dx.$$

The following example brings these formulae to life.

Example 4.6 *Determine the half range sine and cosine series for the function*

$$f(t) = t^2 + t, \quad 0 \le t \le \pi.$$

Solution We have previously determined that, for $f(t) = t^2 + t$

$$a_n = \frac{4(-1)^n}{n^2}, \quad b_n = \frac{2}{n}(-1)^n.$$

For this function, the graph is displayed in Fig. 4.5. If we wish $f(t)$ to be even, then $f(t) = f(-t)$ and so

$$a_n = \frac{2}{\pi}\int_0^{\pi}(t^2 + t)\cos(nt)dt, \quad b_n = 0.$$

On the other hand, if we wish $f(t)$ to be odd, we require $f(t) = -f(-t)$, where

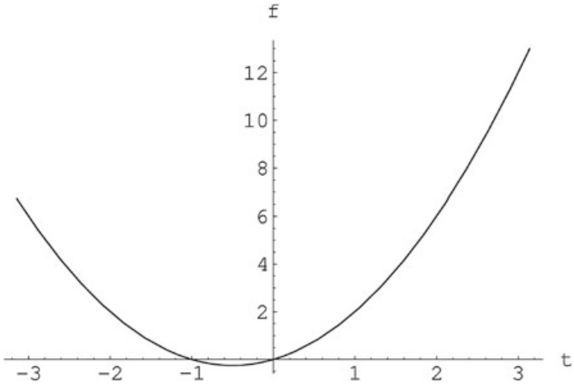


Fig. 4.5 The graph $f(t) = t^2 + t$, shown for the entire range $-\pi \le t \le \pi$ to emphasise that it might be periodic but it is neither odd nor even

$$b'_n = \frac{2}{\pi}\int_0^{\pi}(t^2+t)\sin(nt)dt, \quad a'_n = 0.$$

Both of these series can be calculated quickly using the same trick as used to determine the whole Fourier series, namely, let

$$k = a_n + ib'_n = \frac{2}{\pi}\int_0^{\pi}(t^2+t)e^{int}dt.$$

We evaluate this carefully using integration by parts and show the details.

$$\begin{aligned}
k &= \frac{2}{\pi}\int_0^{\pi}(t^2+t)e^{int}dt \\
&= \frac{2}{\pi}\left[\frac{t^2+t}{in}e^{int}\right]_0^{\pi} - \frac{2}{\pi}\int_0^{\pi}\frac{2t+1}{in}e^{int}dt \\
&= \frac{2}{\pi}\left[\frac{t^2+t}{in}e^{int} - \frac{2t+1}{(in)^2}e^{int} + \frac{2}{(in)^3}e^{int}\right]_0^{\pi} \\
&= \frac{2}{\pi}\left[-\frac{\pi^2+\pi}{in} + \frac{2\pi+1}{n^2} + \frac{2i}{n^3}\right](-1)^n \\
&- \frac{2}{\pi}\left[-\frac{1}{n^2} + \frac{2i}{n^3}\right]
\end{aligned}$$

from which

$$a_n = \frac{2}{\pi}\left[\frac{2\pi+1}{n^2}(-1)^n + \frac{1}{n^2}\right]$$

and

$$b'_n = \frac{2}{\pi}\left[-\frac{(\pi^2+\pi)}{n}(-1)^n + \frac{2}{n^3}((-1)^n - 1)\right].$$

The constant term $\frac{1}{2}a_0$ is given by

$$\begin{aligned} a_0 &= \frac{2}{\pi}\int_0^{\pi}(t^2+t)dt \\ &= \frac{2}{\pi}\left[\frac{1}{3}t^3+\frac{1}{2}t^2\right]_0^{\pi} \\ &= 2\left[\frac{\pi^2}{3}+\frac{\pi}{2}\right] \end{aligned}$$

so the constant term in the Fourier series ($a_0/2$) is

$$\left[\frac{\pi^2}{3}+\frac{\pi}{2}\right].$$

Therefore we have deduced the following two series: the *even* series for t^2+t:-

$$\left[\frac{\pi^2}{3}+\frac{\pi}{2}\right]+\frac{2}{\pi}\sum_{n=1}^{\infty}\left[\frac{2\pi+1}{n^2}(-1)^n+\frac{1}{n^2}\right]\cos(nt), \quad x\in(0,\pi)$$

and the *odd* series for t^2+t

$$\frac{2}{\pi}\sum_{n=1}^{\infty}\left[\frac{(\pi^2+\pi)}{n}(-1)^{n+1}+\frac{2}{n^3}((-1)^n-1)\right]\sin(nt), \quad x\in(0,\pi).$$

They are pictured in Fig. 4.6. To frame this in terms of the theory of linear spaces, the originally chosen set of basis functions

$$\frac{1}{\sqrt{2}}, \cos(x), \sin(x), \cos(2x), \sin(2x), \ldots$$

is no longer a basis in the halved interval $[0,\pi]$. However the sequences

$$\frac{1}{\sqrt{2}}, \cos(x), \cos(2x), \ldots$$

and

$$\sin(x), \sin(2x), \sin(3x)\ldots$$

are, separately, both bases. Half range series are thus legitimate.

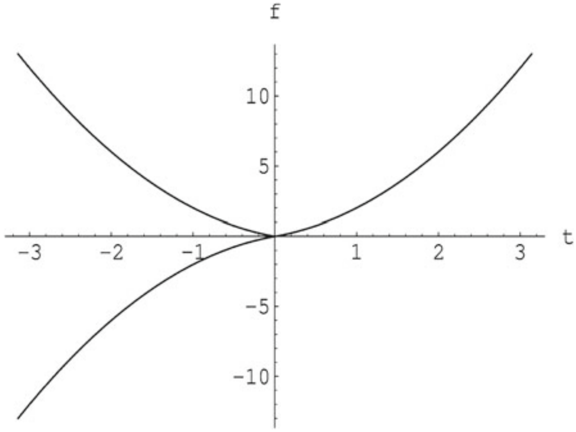


Fig. 4.6 The function $f(t)$ displayed both as an even function and as an odd function

4.6 Properties of Fourier Series

In this section we shall be concerned with the integration and differentiation of Fourier series. Intuitively, it is the differentiation of Fourier series that poses more problems than integration. This is because differentiating $\cos(nx)$ or $\sin(nx)$ with respect to x gives $-n\sin(nx)$ or $n\cos(nx)$ which for large n are both larger in magnitude than the original terms. Integration on the other hand gives $\sin(nx)/n$ or $-\cos(nx)/n$, both smaller in magnitude. For those familiar with numerical analysis this comes as no surprise as numerical differentiation always needs more care than numerical integration which by comparison is safe. The following theorem covers the differentiation of Fourier series.

Theorem 4.5 *If f is continuous on $[-\pi, \pi]$ and piecewise differentiable in $(-\pi, \pi)$ which means that the derivative f' is piecewise continuous on $[-\pi, \pi]$, and if $f(x)$ has the Fourier series*

$$f(x) \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\{a_n \cos(nx) + b_n \sin(nx)\}$$

then the Fourier series of the derivative of $f(x)$ is given by

$$f'(x) \sim \sum_{n=1}^{\infty}\{-na_n \sin(nx) + nb_n \cos(nx)\}.$$

The proof of this theorem follows standard analysis and is not given here. The integration of a Fourier series poses less of a problem and can virtually always take place. The conditions for a function $f(x)$ to possess a Fourier series are similar to those required for integrability (piecewise continuity is sufficient), therefore integrating term by term can occur. A minor problem arises because the result is not necessarily another Fourier series. A term linear in x is produced by integrating the constant term whenever this is not zero. Formally, the following theorem covers the integration of Fourier series. It is not proved either, although a related more general result is derived a little later as a precursor to Parseval's theorem.

Theorem 4.6 *If f is piecewise continuous on the interval $[-\pi, \pi]$ and has the Fourier series*

$$f(x) \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\{a_n \cos(nx) + b_n \sin(nx)\}$$

then for each $x \in [-\pi, \pi]$,

$$\int_{-\pi}^{x} f(t)dt = \frac{a_0(x+\pi)}{2} + \sum_{n=1}^{\infty}\left[\frac{a_n}{n}\sin(nx) - \frac{b_n}{n}(\cos(nx) - \cos(n\pi))\right]$$

and the function on the right converges uniformly to the function on the left.

Let us discuss the details of differentiating and integrating Fourier series via the three series for the functions x^3, x^2 and x in the range $[-\pi, \pi]$. Formally, these three functions can be rendered periodic of period 2π by demanding that for each, $f(x) = f(x+2\pi)$ and using Theorem 4.3 at the end points. The three Fourier series themselves can be derived using Eq. 4.4 and are

$$x^3 \sim \sum_{n=1}^{\infty}(-1)^n \frac{2}{n^3}(6 - \pi^2 n^2)\sin(nx)$$

$$x^2 \sim \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4(-1)^n}{n^2}\cos(nx)$$

$$x \sim \sum_{n=1}^{\infty}\frac{2}{n}(-1)^{n+1}\sin(nx)$$

all valid for $-\pi < x < \pi$. We state without proof the following facts about these three series. The series for x^2 is uniformly convergent. Neither the series for x nor that for x^3 are uniformly convergent. All the series are pointwise convergent. It is therefore legal to differentiate the series for x^2 but not either of the other two. All the series can be integrated. Let us perform the operations and verify these claims. It is certainly true that the term by term differentiation of the series

$$x^2 \sim \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4(-1)^n}{n^2}\cos(nx)$$

gives

$$2x \sim \sum_{n=1}^{\infty} \frac{4}{n}(-1)^{n+1} \sin(nx)$$

which is the same as the Fourier series for x apart from the trivial factor of 2. Integrating a Fourier series term by term leads to the generation of an arbitrary constant. This can only be evaluated by the insertion of a particular value of x. To see how this works, let us integrate the series for x^2 term by term. The result is

$$\frac{x^3}{3} \sim \frac{\pi^2}{3}x + A + \sum_{n=1}^{\infty} \frac{4}{n^3}(-1)^n \sin(nx)$$

where A is an arbitrary constant. Putting $x = 0$ gives $A = 0$ and inserting the Fourier series for x in place of the x on the right hand side regains the Fourier series for x^3 already given. This integration of Fourier series is not always productive. Integrating the series for x term by term is not useful as there is no easy way of evaluating the arbitrary constant that is generated (unless one happens to know the value of some obscure series). Note also that blindly (and *illegally*) differentiating the series for x^3 or x term by term give nonsense in both cases.

Let us now derive a more general result involving the integration of Fourier series. Suppose $F(t)$ is piecewise differentiable in the interval $(-\pi, \pi)$ and therefore continuous on the interval $[-\pi, \pi]$. Let $F(t)$ be represented by the Fourier series

$$F(t) \sim \frac{A_0}{2} + \sum_{n=1}^{\infty} \left(A_n \cos(nt) + B_n \sin(nt)\right), \quad -\pi < t < \pi,$$

and the usual periodicity

$$F(t) = F(t + 2\pi).$$

Now suppose that we can define another function $G(x)$ through the relationship

$$\int_{-\pi}^{t} G(x)dx = \frac{1}{2}a_0 t + F(t).$$

We then set ourselves the task of determining the Fourier series for $G(x)$. Using that $F(t)$ has a full range Fourier series we have

$$A_n = \frac{1}{\pi}\int_{-\pi}^{\pi} F(t)\cos(nt)dt, \text{ and } B_n = \frac{1}{\pi}\int_{-\pi}^{\pi} F(t)\sin(nt)dt.$$

By the fundamental theorem of the calculus we have that

$$F'(t) = G(t) - \frac{1}{2}a_0.$$

Now, evaluating the integral for A_n by parts once gives

$$A_n = -\frac{1}{n\pi}\int_{-\pi}^{\pi} F'(t)\sin(nt)dt,$$

and similarly,

$$B_n = -\frac{1}{n\pi}\int_{-\pi}^{\pi} F'(t)\cos(nt)dt.$$

Hence, writing the Fourier series for $G(x)$ as

$$G(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos(nx) + b_n\sin(nx)\right), \quad -\pi < x < \pi,$$

and comparing with

$$G(t) = \frac{1}{2}a_0 + F'(t)$$

gives

$$A_n = -\frac{b_n}{n} \text{ and } B_n = \frac{a_n}{n}.$$

We now need to consider the values of the Fourier series for $F(t)$ at the end points $t = -\pi$ and $t = \pi$. With

$$t = \pi,\ F(\pi) = -\frac{a_0\pi}{2} + \int_{-\pi}^{\pi} G(x)dx$$

and with

$$t = -\pi,\ F(-\pi) = \frac{a_0\pi}{2}.$$

Since $F(t)$ is periodic of period 2π we have

$$F(-\pi) = F(\pi) = \frac{1}{2}a_0\pi.$$

Also

$$F(t) \sim \frac{A_0}{2} + \sum_{n=1}^{\infty}\left(-\frac{b_n}{n}\cos(nt) + \frac{a_n}{n}\sin(nt)\right).$$

Putting $t = \pi$ is legitimate in the limit as there is no jump discontinuity at this point (piecewise differentiability), and gives

$$\frac{1}{2}a_0\pi = \frac{1}{2}A_0 - \sum_{n=1}^{\infty}\frac{b_n}{n}\cos(n\pi) = \frac{1}{2}A_0 - \sum_{n=1}^{\infty}(-1)^n\frac{b_n}{n}.$$

We need to relate A_0 to a_0 and the b_n terms. To do this, we use the new form of $F(t)$ namely

$$F(t) \sim \frac{1}{2}a_0\pi + \sum_{n=1}^{\infty}\frac{1}{n}\left[b_n\left((-1)^n - \cos(nt)\right) + a_n\sin(nt)\right]$$

which is written in terms of the Fourier coefficients of $G(t)$, the integral of $F(t)$. To determine the form of this integral we note that

$$\begin{aligned}\int_{-\pi}^{t} G(x)dx &= \frac{1}{2}a_0 t + F(t)\\ &= \frac{1}{2}a_0(\pi + t) + \sum_{n=1}^{\infty}\frac{1}{n}\left[b_n(-1)^n - b_n\cos(nt) + a_n\sin(nt)\right]\end{aligned}$$

Also

$$\int_{-\pi}^{\xi} G(x)dx = \frac{1}{2}a_0(\pi + \xi) + \sum_{n=1}^{\infty}\frac{1}{n}\left[b_n(-1)^n - b_n\cos(n\xi) + a_n\sin(n\xi)\right]$$

so subtracting gives

$$\int_{\xi}^{t} G(x)dx = \frac{1}{2}a_0(t-\xi)+\sum_{n=1}^{\infty}\frac{1}{n}\left[b_n\left(\cos(n\xi) - \cos(nt)\right) + a_n\left(\sin(nt) - \sin(n\xi)\right)\right]$$

which tells us the form of the integral of a Fourier series. Here is an example where the ability to integrate a Fourier series term by term proves particularly useful.

Example 4.7 *Use the Fourier series*

$$x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos(nx)$$

to deduce the value of the series

$$\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{(2k-1)^3}.$$

Solution Utilising the result just derived on the integration of Fourier series, we put $\xi = 0$ and $t = \pi/2$ so we can write

$$\int_0^{\frac{\pi}{2}} x^2 dx = \frac{\pi^2}{3}\left(\frac{\pi}{2} - 0\right) + 4\sum_{n=1}^{\infty}\frac{1}{n}\left[\frac{(-1)^n}{n^2}\sin\left(\frac{n\pi}{2}\right)\right]$$

so

$$\frac{\pi^3}{24} = \frac{\pi^3}{6} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3} \sin\left(\frac{n\pi}{2}\right).$$

Now, since $\sin(n\pi/2)$ takes the values $0, 1, 0, -1, \ldots$ for $n = 0, 1, 2, \ldots$ we immediately deduce that

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3} \sin\left(\frac{n\pi}{2}\right) = -\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(2k-1)^3}$$

which gives

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3} \sin\left(\frac{n\pi}{2}\right) = \frac{1}{4}\left(\frac{\pi^3}{6} - \frac{\pi^3}{24}\right) = \frac{\pi^3}{32}.$$

Whence, putting $n = 2k - 1$ gives the result

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(2k-1)^3} = \frac{\pi^3}{32}.$$

The following theorem can now be deduced.

Theorem 4.7 *If $f(t)$ and $g(t)$ are continuous in $(-\pi, \pi)$ and provided*

$$\int_{-\pi}^{\pi} |f(t)|^2 dt < \infty \ \ \textit{and} \ \ \int_{-\pi}^{\pi} |g(t)|^2 dt < \infty,$$

if $a_n,\ b_n$ are the Fourier coefficients of $f(t)$ and α_n, β_n those of $g(t)$, then

$$\int_{-\pi}^{\pi} f(t)g(t)dt = \frac{1}{2}\pi a_0 \alpha_0 + \pi \sum_{n=1}^{\infty} (\alpha_n a_n + \beta_n b_n).$$

Proof Since

$$f(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(nt) + b_n \sin(nt))$$

and

$$g(t) \sim \frac{\alpha_0}{2} + \sum_{n=1}^{\infty} (\alpha_n \cos(nt) + \beta_n \sin(nt))$$

we can write

$$f(t)g(t) \sim \frac{1}{2}a_0 g(t) + \sum_{n=1}^{\infty} \left(a_n g(t)\cos(nt) + b_n g(t)\sin(nt)\right).$$

Integrating this series from $-\pi$ to π gives

$$\begin{aligned}\int_{-\pi}^{\pi} f(t)g(t)dt &= \frac{1}{2}a_0 \int_{-\pi}^{\pi} g(t)dt \\ &+ \sum_{n=1}^{\infty} \left\{a_n \int_{-\pi}^{\pi} g(t)\cos(nt)dt + b_n \int_{-\pi}^{\pi} g(t)\sin(nt)dt\right\}\end{aligned}$$

provided the Fourier series for $f(t)$ is uniformly convergent, enabling the summation and integration operations to be interchanged. This follows from the Cauchy–Schwarz inequality since

$$\int_{-\pi}^{\pi} |f(t)g(t)|dt \le \left(\int_{-\pi}^{\pi} |f(t)|^2 dt\right)^{1/2} \left(\int_{-\pi}^{\pi} |g(t)|^2 dt\right)^{1/2} < \infty.$$

However, we know that

$$\begin{aligned}\frac{1}{\pi}\int_{-\pi}^{\pi} g(t)\cos(nt)dt &= \alpha_n \\ \frac{1}{\pi}\int_{-\pi}^{\pi} g(t)\sin(nt)dt &= \beta_n\end{aligned}$$

and that

$$\alpha_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} g(t)dt$$

so this implies

$$\int_{-\pi}^{\pi} f(t)g(t)dt = \frac{1}{2}\pi a_0 \alpha_0 + \pi \sum_{n=1}^{\infty} (\alpha_n a_n + \beta_n b_n)$$

as required. □

If we put $f(t) = g(t)$ in the above result, the following important theorem immediately follows.

Theorem 4.8 (Parseval) *If $f(t)$ is continuous in the range $(-\pi, \pi)$, is square integrable (i.e. $\int_{-\pi}^{\pi} [f(t)]^2 dt < \infty$) and has Fourier coefficients a_n, b_n then*

$$\int_{-\pi}^{\pi}[f(t)]^2 dt = 2\pi a_0^2 + \pi\sum_{n=1}^{\infty}(a_n^2 + b_n^2).$$

Proof This is immediate from Theorem 4.7 by putting

$$f(t) = g(t)$$

□

This is a useful result for mathematicians, but perhaps its most helpful attribute lies in its interpretation. The left hand side represents the mean square value of $f(t)$ (once it is divided by 2π). It can therefore be thought of in terms of energy if $f(t)$ represents a signal. What Parseval's theorem states therefore is that the energy of a signal expressed as a waveform is proportional to the sum of the squares of its Fourier coefficients. For now, let us content ourselves with a mathematical consequence of the theorem.

Example 4.8 *Given the Fourier series*

$$t^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos(nt)$$

deduce the value of

$$\sum_{n=1}^{\infty}\frac{1}{n^4}.$$

Solution Applying Parseval's theorem to this series, the left hand side becomes

$$\int_{-\pi}^{\pi}(t^2)^2 dt = \frac{2}{5}\pi^5.$$

The right hand side becomes

$$\pi\left(\frac{\pi^2}{3}\right)^2 + \pi\sum_{n=1}^{\infty}\frac{16}{n^4}.$$

Equating these leads to

$$\frac{2}{5}\pi^5 = \frac{2}{9}\pi^5 + \pi\sum_{n=1}^{\infty}\frac{16}{n^4}$$

or

$$16\sum_{n=1}^{\infty}\frac{1}{n^4} = \pi^4\left(\frac{2}{5} - \frac{2}{9}\right).$$

Hence

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$$

4.7 Generalised Fourier Series

To finish off this chapter, let us generalise. Instead of using specific trigonometric functions, we can use any functions provided as a set they form a basis and are orthogonal. Call the functions $\phi_n(t)$ where n is an integer. Then the piecewise continuous function $f(t)$ can be expressed as the generalised Fourier series:

$$f(t) = \sum_{n=1}^{\infty} k_n \phi_n(t). \tag{4.5}$$

The coefficients k_n are retrieved using the orthogonality of the basis functions $\phi_n(t)$ through the integral

$$\int_{t_0}^{t_1} f(t)\phi_n(t)dt = \sum_{m=1}^{\infty} k_m \int_{t_0}^{t_1} \phi_m(t)\phi_n(t)dt$$

and of course the orthogonality of the basis functions $\phi_n(t)$ means that

$$\int_{t_0}^{t_1} \phi_m(t)\phi_n(t)dt = \delta_{m,n}\gamma_n$$

where $\delta_{m,n}$ is the Kronecker delta and γ_n is there as the orthogonal functions ϕ_n are not orthonormal. [For those aware of Cartesian tensors, the summation convention does not apply here; the right hand side is *not* equal to γ_m] Putting $m = n$ gives

$$\gamma_n = \int_{t_0}^{t_1} \phi_n^2(t)dt.$$

As a reminder, Kronecker's delta is defined as follows:

$$\delta_{m,n} = \begin{cases} 0 & m \neq n \\ 1 & m = n \end{cases}.$$

Being careful to change the dummy variable from n to m in the generalised Fourier series, Eq. 4.5, first the coefficients k_n are given in terms of n by

$$k_n = \frac{1}{\gamma_n} \int_{t_0}^{t_1} f(t)\phi_n(t)dt.$$

In all of the above $[t_0, t_1]$ is the interval over which the functions $\phi_n(t)$ are orthogonal. The function $f(t)$ is expressible as a generalised Fourier series for values of t that lie inside this interval. Typically the interval is $[0, 2L]$ but L can be infinite. Let us return to how to choose specific function sets $\phi_n(t)$. Choosing the basic trigonometric functions sine and cosine that are the backbone of Fourier series certainly is a good choice as their properties are familiar. We have already established the criteria needed in terms of forming a basis and orthogonality in terms of linear spaces. The set

$$\left\{ \frac{1}{\sqrt{2}}, \sin(x), \cos(x), \sin(2x), \cos(2x), \ldots \right\}$$

certainly forms a basis (this one's orthonormal) for all periodic piecewise continuous functions on $[-\pi, \pi]$. The one we use

$$\{e_0, e_1, e_2, \ldots\} = \left\{ \frac{1}{2}, \sin(x), \cos(x), \sin(2x), \cos(2x), \ldots \right\}$$

fails to be orthonormal because using the inner product of Theorem 4.4 $e_0 = 1/2$ and $\langle e_0, e_0 \rangle = \frac{1}{2} \neq 1$, but this is a trivial failure, only there because users of Fourier series like to avoid square roots. All the important orthogonality conditions are valid. The real question is what makes sine and cosine so special, and are there other bases that would do as well? To lead us to the answer, note that the second order differential equation

$$\frac{d^2x}{dt^2} + n^2 x = 0$$

has general solution

$$x = A\cos(nt) + B\sin(nt).$$

If we consider a more general second order differential equation that contains a parameter that can take integer values, then we could label one solution of this equation $\phi_n(t)$. On the other hand there's no earthly reason why $\phi_n(t)$ needs to satisfy a differential equation provided there is a basis for all piecewise continuous functions. The reason we pick solutions of differential equations is that these turn out to be a guaranteed source of bases. Solutions of second order differential equations that have a parameter associated with them that can be an integer almost always can be used to generate the linear space for all piecewise continuous functions. The mathematics behind this was done in the 19th century by such luminaries as J. C. F. Sturm (1803–1855) and J. Liouville (1808–1882). Such pure mathematics is however beyond the scope of this text. Let us merely state a couple of these second order differential equations as examples of specific choices for $\phi_n(t)$. Perhaps the best known examples are Legendre's equation

$$(1-t^2)\frac{d^2P_n}{dt^2} - 2t\frac{dP_n}{dt} + n(n+1)P_n = 0, \quad n = 0, 1, \ldots$$

and Bessel's equation

$$t^2\frac{d^2J_n}{dt^2} + t\frac{dJ_n}{dt} + (t^2 - n^2)J_n = 0 \quad n = 0, 1 \ldots$$

The generalised series are termed Fourier-Legendre and Fourier-Bessel series respectively. All the worries about convergence etc. are allayed as these special functions as they are called are all appropriately well behaved. The Fourier Legendre series is

$$f(t) = \sum_{n=1}^{\infty} k_n P_n(t)$$

with

$$\int_{-1}^{1} P_m(t)P_n(t)dt = \frac{2}{2m+1}\delta_{m,n}$$

so

$$k_n = \frac{2n+1}{2}\int_{-1}^{1} P_n(t)f(t)dt.$$

The Fourier-Bessel series is a bit different as the series is based on where the zeros of the Bessel function are rather than the order of the Bessel function itself. These in turn depend on the order parameter n. For this reason we call the Bessel function $J_r(t)$ so that n is free to be our dummy variable. Suppose the zeros of $J_r(t)$ are labelled $\alpha_n, n = 1, 2, 3, \ldots$ so that $J_r(\alpha_n) = 0$ for all integers n. The Fourier-Bessel equation is then

$$f(t) = \sum_{n=1}^{\infty} k_n J_r(t\alpha_n).$$

The orthogonality condition is also a bit different. First of all

$$\int_0^1 tf(t)J_r(t\alpha_n)dt = \sum_{n=1}^{\infty} k_m \int_0^1 tJ_r(t\alpha_n)J_r(t\alpha_m)dt$$

and the condition itself is

$$\int_0^1 tJ_r(t\alpha_n)J_r(t\alpha_m)dt = \frac{1}{2}\delta_{m,n}[J_{r+1}(\alpha_m)]^2$$

so that

$$\int_0^1 tf(t)J_r(t\alpha_n)dt = \frac{1}{2}k_n[J_{r+1}(\alpha_n)]^2$$

with

$$k_n = \frac{2}{[J_{r+1}(\alpha_n)]^2}\int_0^1 tf(t)J_r(t\alpha_n)dt.$$

The extra t under the integral sign is called a *weight function.* Let's leave these specific special series here as knowledge of the properties of $P_n(t)$ and $J_r(t\alpha_n)$ are required.

Before doing an example based on a different special function it is worth stating the generalised version of Parseval's theorem

$$\int_{t_0}^{t_1} f^2(t)dt = \sum_{n=1}^{\infty}\gamma k_n^2.$$

Example 4.9 *Use the equation*

$$\frac{d}{dt}\left[te^{-t}\frac{dL_n}{dt}\right] + ne^{-t}L_n = 0$$

to show that $L_m(t), L_n(t); m, n = 0, 1, 2, 3\ldots$ *obey the orthogonality relation*

$$\int_0^{\infty} e^{-t}L_m(t)L_n(t)dt = 0$$

($L_n(t)$ *is a Laguerre function, and the differential equation*

$$t\frac{d^2y}{dt^2} + (1-t)\frac{dy}{dt} + ny = 0$$

is called Laguerre's differential equation.)

Solution Start with the two equations

$$\frac{d}{dt}\left[te^{-t}\frac{dL_n}{dt}\right] + ne^{-t}L_n = 0$$

and

$$\frac{d}{dt}\left[te^{-t}\frac{dL_m}{dt}\right] + me^{-t}L_m = 0$$

take L_m times the first minus L_n times the second and the solution follows after integration by parts. Here are some details; first of all we get:

$$L_m \frac{d}{dt}\left[te^{-t}\frac{dL_n}{dt}\right] - L_n \frac{d}{dt}\left[te^{-t}\frac{dL_m}{dt}\right] - (m-n)e^{-t}L_m L_n = 0.$$

Now integrate this between 0 and ∞. The first two terms are evaluated as follows

$$\left[L_m te^{-t}\frac{dL_n}{dt} - L_n te^{-t}\frac{dL_m}{dt}\right]_0^\infty - \int_0^\infty \left[\frac{dL_m}{dt}te^{-t}\frac{dL_n}{dt} - \frac{dL_n}{dt}te^{-t}\frac{dL_m}{dt}\right]dt.$$

This is zero as the first integrated part vanishes at both limits (L_n is a polynomial so $e^{-t}L_n$ tends to zero as $t \to \infty$), and the integrand on the second part is zero. Hence

$$\int_0^\infty e^{-t}L_m(t)L_n(t)dt = 0$$

provided $m \neq n$. If $m = n$ the integral is non-zero and not unity. So L_m and L_n are orthogonal but not orthonormal, this is quite typical.

Note the presence of e^{-t} in the integrand. It is the *weight function* for the Laguerre functions. The last example provides an introduction to Sturm-Liouville theory which uses the orthogonality of the solutions to homogeneous ordinary differential equations to find more general solutions to non-homogeneous differential equations when put in the form of an eigenvalue problem (see texts on Ordinary Differential Equations, for example King, Billingham and Otto (2003)).

A final word on generalised Fourier series. If a function $f(t)$ is expressed as a generalised Fourier series it does not have to be periodic. The range dictated by the orthogonality condition, for example $[-1, 1]$ for the Fourier-Legendre series or $[0, 1]$ for the Fourier-Bessel series tells you the interval over which the function $f(t)$ has to be defined, but there is no implied periodicity as there is for Fourier series. The periodicity of these is solely due to the periodicity of sine and cosine. There can be a pattern in the generalised series due to the properties of the particular orthogonal function, one example of this is displayed through the function $f(t)$ in exercise 16 below. In this exercise, $f(t)$ is not periodic but $f(\ln t)$ *is* periodic.

4.8 Exercises

1. Use the Riemann–Lebesgue lemma to show that

$$\lim_{m\to\infty}\int_0^\pi g(t)\sin\left(m+\frac{1}{2}\right)t\,dt = 0,$$

where $g(t)$ is piecewise continuous on the interval $[0, \pi]$.

2. $f(t)$ is defined by

$$f(t) = \begin{cases} t & 0 \le t \le \frac{1}{2}\pi \\ \frac{1}{2}\pi & \frac{1}{2}\pi \le t \le \pi \\ \pi - \frac{1}{2}t & \pi \le t \le 2\pi. \end{cases}$$

Sketch the graph of $f(t)$ and determine a Fourier series assuming $f(t) = f(t + 2\pi)$.

3. Determine the Fourier series for $f(x) = H(x)$, the Heaviside unit step function, in the range $[-\pi, \pi]$, $f(x) = f(x + 2\pi)$. Hence find the value of the series

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots .$$

4. Find the Fourier series of the function

$$f(x) = \begin{cases} \sin(\frac{1}{2}x) & 0 \le x \le \pi \\ -\sin(\frac{1}{2}x) & \pi < x \le 2\pi \end{cases}$$

with $f(x) = f(x + 2\pi)$.

5. Determine the Fourier series for the function $f(x) = 1 - x^2$, $f(x) = f(x + 2\pi)$. Suggest possible values of $f(x)$ at $x = \pi$.
6. Deduce that the Fourier series for the function $f(x) = e^{ax}$, $-\pi < x < \pi$, a a real number is

$$\frac{\sinh(\pi a)}{\pi} \left\{ \frac{1}{a} + 2\sum_{n=1}^{\infty} \frac{(-1)^n}{a^2 + n^2} (a\cos(nx) - n\sin(nx)) \right\}.$$

Hence find the values of the four series:

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{a^2 + n^2}, \quad \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{a^2 + n^2}, \quad \sum_{n=1}^{\infty} \frac{1}{a^2 + n^2}, \quad \sum_{n=-\infty}^{\infty} \frac{1}{a^2 + n^2}.$$

7. If

$$f(t) = \begin{cases} -t + e^t & -\pi \le t < 0 \\ t + e^t & 0 \le t < \pi \end{cases}$$

where $f(t) = f(t + 2\pi)$, sketch the graph of $f(t)$ for $-4\pi \le t \le 4\pi$ and obtain a Fourier series expansion for $f(t)$.

8. Find the Fourier series expansion of the function $f(t)$ where

$$f(t) = \begin{cases} \pi^2 & -\pi < t < 0 \\ (t - \pi)^2 & 0 \le t < \pi \end{cases}$$

and $f(t) = f(t + 2\pi)$. Hence determine the values of the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \text{ and } \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}.$$

9. Determine the two Fourier half-range series for the function $f(t)$ defined in Exercise 8, and sketch the graphs of the function in both cases over the range $[-2\pi \le t \le 2\pi]$.
10. Given the half range sine series

$$t(\pi - t) = \frac{8}{\pi} \sum_{n=1}^{\infty} \frac{\sin(2n-1)t}{(2n-1)^3}, \quad 0 \le t \le \pi$$

use Parseval's Theorem to deduce the value of the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(2n-1)^6}$.

Hence deduce the value of the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^6}$.
11. Deduce that the Fourier series for the function $f(x) = x^4, \quad -\pi < x < \pi$, is

$$x^4 \sim \frac{\pi^4}{5} + \sum_{n=1}^{\infty} \frac{8(-1)^n}{n^4}(\pi^2 n^2 - 6)\cos(nx).$$

Explain why this series contains no sine terms. Use this series to find the value of the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4}$$

given that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}.$$

Assuming (correctly) that this Fourier series is uniformly convergent, use it to derive the Fourier series for x^3 over the range $-\pi < x < \pi$.
12. Given the Fourier series

$$x \sim \sum_{n=1}^{\infty} \frac{2}{n}(-1)^{n+1} \sin(nx), \quad -\pi < x < \pi,$$

integrate term by term to obtain the Fourier series for x^2, evaluating the constant of integration by integrating both sides over the range $[-\pi, \pi]$. Use the same integration technique on the Fourier series for x^4 given in the last exercise to deduce the Fourier series for x^5 over the range $-\pi < x < \pi$.
13. In an electrical circuit, the voltage is given by the "top-hat" function

$$V(t) = \begin{cases} 40 & 0 < t < 2 \\ 0 & 2 < t < 5. \end{cases}$$

Obtain the first five terms of the complex Fourier series for $V(t)$.

14. Show that the functions $P_n(t)$ that satisfy the ordinary differential equation

$$(1 - t^2)\frac{d^2 P_n}{dt^2} - 2t\frac{dP_n}{dt} + n(n+1)P_n = 0, \quad n = 0, 1, \dots$$

are orthogonal in the interval $[-1, 1]$, where the inner product is defined by

$$\langle p, q \rangle = \int_{-1}^{1} pq\,dt.$$

15. Show that $y = \phi_n(t) = \sin(n \ln t)$ is a solution to the ordinary differential equation

$$\left[t\frac{d}{dt}\left(t\frac{dy}{dt}\right)\right] + n^2 y = 0.$$

Hence show that the functions $\phi_n(t)$, $n = 1, 2, \dots$ are orthogonal in the range $[1, e^\pi]$ with weight function $1/t$.

16. Suppose $g(x) = g(x + L)$ is a periodic piecewise continuous function defined for real variable x. Define $x = \ln t$ and let $g(x) = g(\ln t) = f(t)$ have a half range Fourier series

$$g(x) = \sum_{n=1}^{\infty} b_n \sin(nx)$$

over the range $[0, \pi]$. Using this half range series, find the generalised Fourier series for the function:

$$f(t) = \begin{cases} 1 \; 1 \le t < e^\pi \\ 0 \text{ otherwise} \end{cases}$$

based on the generalised series $\phi_n(x)$ introduced in exercise 15.

Chapter 5
Partial Differential Equations

5.1 Introduction

In previous chapters, we have explained how ordinary differential equations can be solved using Laplace transforms. In Chap. 4, Fourier series were introduced, and the important property that any reasonable function can be expressed as a Fourier series derived. In this chapter, these ideas are brought together, and the solution of certain types of partial differential equation using both Laplace transforms and Fourier series are explored. The study of the solution of partial differential equations (abbreviated PDEs) is a vast topic that it is neither possible nor appropriate to cover in a single chapter. There are many excellent texts (Weinberger (1965), Sneddon (1957) and Williams (1980) to name but three) that have become standard. Here we shall only be interested in certain types of PDE that are amenable to solution by Laplace transform.

Of course, to start with we will have to assume you know something about partial derivatives. If a function depends on more than one variable, then it is in general possible to differentiate it with respect to one of them provided all the others are held constant while doing so. Thus, for example, a function of three variables $f(x, y, z)$ (if differentiable in all three) will have three derivatives written

$$\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \text{ and } \frac{\partial f}{\partial z}.$$

The three definitions are straightforward and, hopefully familiar.

$$\frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \left\{ \frac{f(x + \Delta x, y, z) - f(x, y, z)}{\Delta x} \right\}$$

y and z are held constant,

$$\frac{\partial f}{\partial y} = \lim_{\Delta y \to 0} \left\{ \frac{f(x, y + \Delta y, z) - f(x, y, z)}{\Delta y} \right\}$$

P. Dyke, *An Introduction to Laplace Transforms and Fourier Series*,
Springer Undergraduate Mathematics Series, DOI: 10.1007/978-1-4471-6395-4_5,

x and z are held constant, and

$$\frac{\partial f}{\partial z} = \lim_{\Delta z \to 0} \left\{ \frac{f(x, y, z + \Delta z) - f(x, y, z)}{\Delta z} \right\}$$

x and y are held constant. If all this is deeply unfamiliar, mysterious and a little terrifying, then a week or two with an elementary text on partial differentiation is recommended. It is an easy task to perform: simply differentiate with respect to one of the variables whilst holding the others constant. Also, it is easy to deduce that all the normal rules of differentiation apply as long it is remembered which variables are constant and which is the one to which the function is being differentiated. One example makes all this clear.

Example 5.1 *Find all first order partial derivatives of the functions (a)* x^2yz*, and (b)* $x \sin(x + yz)$.

Solution (a) The partial derivatives are as follows:-

$$\frac{\partial}{\partial x}(x^2yz) = 2xyz$$

$$\frac{\partial}{\partial y}(x^2yz) = x^2z$$

and

$$\frac{\partial}{\partial z}(x^2yz) = x^2y.$$

(b) The partial derivatives are as follows:-

$$\frac{\partial}{\partial x}(x \sin(x + yz)) = \sin(x + yz) + x \cos(x + yz)$$

which needs the product rule,

$$\frac{\partial}{\partial y}(x \sin(x + yz)) = xz \cos(x + yz)$$

and

$$\frac{\partial}{\partial z}(x \sin(x + yz)) = xy \cos(x + yz)$$

which do not.

There are chain rules for determining partial derivatives when f is a function of u, v and w which in turn are functions of x, y and z

$$u = u(x, y, z), \;\; v = v(x, y, z) \text{ and } w = w(x, y, z).$$

This is direct extension of the "function of a function" rule for single variable differentiation. There are other new features such as the Jacobian. We shall not pursue these here; instead the interested reader is referred to specialist texts such as Weinberger (1965) or Zauderer (1989).

5.2 Classification of Partial Differential Equations

In this book, we will principally be concerned with those partial differential equations that can be solved using Laplace transforms, perhaps with the aid of Fourier series. Thus we will eventually concentrate on second order PDEs of a particular type. However, in order to place these in context, we need to quickly review (or introduce for those readers new to this subject) the three different generic types of second order PDE.

The general second order PDE can be written

$$a_1\frac{\partial^2\phi}{\partial x^2}+b_1\frac{\partial^2\phi}{\partial x\partial y}+c_1\frac{\partial^2\phi}{\partial y^2}+d_1\frac{\partial\phi}{\partial x}+e_1\frac{\partial\phi}{\partial y}+f_1\phi=g_1 \tag{5.1}$$

where $a_1, b_1, c_1, d_1, e_1, f_1$ and g_1 are suitably well behaved functions of x and y. However, this is not a convenient form of the PDE for ϕ. By judicious use of Taylor's theorem and simple co-ordinate transformations it can be shown (e.g. Williams (1980), Chap. 3) that there are *three* basic types of linear second order partial differential equation. These standard types of PDE are termed hyperbolic, parabolic and elliptic following geometric analogies and are referred to as *canonical forms*. A crisp notation we introduce at this point is the *suffix derivative notation* whereby

$$\begin{aligned}
\phi_x &= \frac{\partial\phi}{\partial x}\\
\phi_y &= \frac{\partial\phi}{\partial y},\ \text{etc.}\\
\phi_{xx} &= \frac{\partial^2\phi}{\partial x^2}\\
\phi_{yy} &= \frac{\partial^2\phi}{\partial y^2}\\
\phi_{xy} &= \frac{\partial^2\phi}{\partial x\partial y},\ \text{etc.}
\end{aligned}$$

This notation is very useful when writing large complicated expressions that involve partial derivatives. We use it now for writing down the three canonical forms of second order partial differential equations

$$\begin{array}{ll} \text{hyperbolic} & a_2\phi_{xy} + b_2\phi_x + c_2\phi_y + f_2\phi = g_2 \\ \text{elliptic} & a_3(\phi_{xx} + \phi_{yy}) + b_3\phi_x + c_3\phi_y + f_3\phi = g_3 \\ \text{parabolic} & a_4\phi_{xx} + b_4\phi_x + c_4\phi_y + f_4\phi = g_4. \end{array}$$

In these equations the a's, b's, c's, f's and g's are functions of the variables x, y. Laplace transforms are useful in solving parabolic and some hyperbolic PDEs. They are not in general useful for solving elliptic PDEs.

Let us now turn to some practical examples in order to see how partial differential equations are solved. The commonest hyperbolic equation is the one dimensional wave equation. This takes the form

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$$

where c is a constant called the *celerity* or *wave speed*. This equation can be used to describe waves travelling along a string, in which case u will represent the displacement of the string from equilibrium, x is distance along the string's equilibrium position, and t is time. As anyone who is familiar with string instruments will know, u takes the form of a wave. The derivation of this equation is not straightforward, but rests on the assumption that the displacement of the string from equilibrium is small. This means that x is virtually the distance along the string. If we think of the solution in (x, t) space, then the lines $x \pm ct =$ constant assume particular importance. They are called *characteristics* and the *general* solution to the wave equation has the general form

$$u = f(x - ct) + g(x + ct)$$

where f and g are arbitrary functions. If we expand f and g as Fourier series over the interval $[0, L]$ in which the string exists (for example between the bridge and the top (machine head) end of the fingerboard in a guitar) then it is immediate that u can be thought of as an infinite superposition of sinusoidal waves:-

$$\begin{aligned} u(x, t) &= \sum_{n=0}^{\infty} a_n \cos[n(x - ct)] + b_n \sin[n(x - ct)] \\ &\quad + \sum_{m=0}^{\infty} a'_m \cos[m(x + ct)] + b'_m \sin[m(x + ct)] + \frac{a_0}{\sqrt{2}} + \frac{a'_0}{\sqrt{2}}. \end{aligned}$$

If the boundary conditions are appropriate to a musical instrument, i.e. $u = 0$ at $x = 0, L$ (all t) then this provides a good visual form of a Fourier series.

Although it is possible to use the Laplace transform to solve such wave problems, this is rarely done as there are more natural methods and procedures that utilise the wave-like properties of the solutions but are outside the scope of this text. (What we are talking about here is the *method of characteristics*—see e.g. Williams (1980) Chap. 3.)

There is one particularly widely occurring elliptic partial differential equation which is mentioned here but cannot in general be solved using Laplace transform techniques. This is Laplace's equation which, in its two dimensional form is

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0.$$

Functions ϕ that satisfy this equation are called *harmonic* and possess interesting mathematical properties. Perhaps the most important of these is the following. A function $\phi(x, y)$ which is harmonic in a domain $D \in \mathbb{R}^2$ has its maximum and minimum values on ∂D, the border of D, and not inside D itself. Laplace's equation occurs naturally in the fields of hydrodynamics, electromagnetic theory and elasticity when steady state problems are being solved in two dimensions. Examples include the analysis of standing water waves, the distribution of heat over a flat surface very far from the source a long time after the source has been switched on, and the vibrations of a membrane. Many of these problems are approximations to parabolic or wave problems that *can* be solved using Laplace transforms. There are books devoted to the solutions of Laplace's equation, and the only reason its solution is mentioned here is because the properties associated with harmonic functions are useful in providing checks to solutions of parabolic or hyperbolic equations in some limiting cases. Let us without further ado go on to discuss parabolic equations.

The most widely occurring parabolic equation is called the heat conduction equation. In its simplest one dimensional form (using the two variables t (time) and x (distance)) it is written

$$\frac{\partial \phi}{\partial t} = \kappa \frac{\partial^2 \phi}{\partial x^2}.$$

This equation describes the manner in which heat $\phi(x, t)$ is conducted along a bar made of a homogeneous substance located along the x axis. The thermal conductivity (or thermal diffusivity) of the bar is a positive constant that has been labelled κ. One scenario is that the bar is cold (at room temperature say) and that heat has been applied to a point on the bar. The solution to this equation then describes the manner in which heat is subsequently distributed along the bar. Another possibility is that a severe form of heat, perhaps using a blowtorch, is applied to one point of the bar for a very short time then withdrawn. The solution of the heat conduction equation then shows how this heat gets conducted away from the site of the flame. A third possibility is that the rod is melting, and the equation is describing the way that the interface between the melted and unmelted rod is travelling away from the heat source that is causing the melting. Solving the heat conduction equation would predict the subsequent heat distribution, including the speed of travel of this interface. Each of these problems is what is called an *initial value problem* and this is precisely the kind of PDE that can be solved using Laplace transforms. The bulk of the rest of this chapter is indeed devoted to solving these. One more piece of "housekeeping" is required however, that is the use of Fourier series in enabling boundary conditions to be satisfied. This in turn requires knowledge of the technique of separation of variables. This is probably revision, but in case it is not, the next section is devoted to it.

5.3 Separation of Variables

The technique of separating the variables will certainly be a familiar method for solving ordinary differential equations in the cases where the two variables can be isolated to occur exclusively on either side of the equality sign. In partial differential equations, the situation is slightly more complicated. It can be applied to Laplace's equation and to the wave equation, both of which were met in the last section. However, here we solve the heat conduction equation.

We consider the problem of solving the heat conduction equation together with the boundary conditions as specified below:

$$\begin{aligned} \frac{\partial \phi}{\partial t} &= \kappa \frac{\partial^2 \phi}{\partial x^2}, \quad x \in [0, L] \\ \phi(x, 0) &= f(x) \;\; \text{at time} \quad t = 0 \\ \phi(0, t) &= \phi(L, t) = 0 \;\; \text{for all time.} \end{aligned}$$

The fundamental assumption for separating variables is to let

$$\phi(x, t) = T(t)X(x)$$

so that the heat conduction equation becomes

$$T'X = \kappa T X''$$

where prime denotes the derivative with respect to t or x. Dividing by XT we obtain

$$\frac{T'}{T} = \kappa \frac{X''}{X}.$$

The next step is crucial to understand. The left hand side is a function of t only, and the right hand side is a function of x only. *As t and x are independent variables, these must be equal to the same constant.* This constant is called the separation constant. It is wise to look ahead a little here. As the equation describes the very real situation of heat conduction, we should look for solutions that will decay as time progresses. This means that T is likely to decrease with time which in turn leads us to designate the separation constant as negative. Let it be $-\alpha^2$, so

$$\frac{T'}{T} = -\alpha^2 \text{ giving } T(t) = T_0 e^{-\alpha^2 t}, \qquad t \geq 0,$$

and
$$\frac{X''}{X} = -\frac{\alpha^2}{\kappa} \text{ giving } X(x) = a' \cos\left(\frac{\alpha x}{\sqrt{\kappa}}\right) + b' \sin\left(\frac{\alpha x}{\sqrt{\kappa}}\right).$$

Whence the solution is

$$\phi(x,t) = e^{-\alpha^2 t}\left(a\cos\left(\frac{\alpha x}{\sqrt{\kappa}}\right) + b\sin\left(\frac{\alpha x}{\sqrt{\kappa}}\right)\right).$$

At time $t = 0$, $\phi(x, 0) = f(x)$ which is some prescribed function of x (the initial temperature distribution along a bar $x \in [0, L]$ perhaps) then we would seem to require that
$$f(x) = a\cos\left(\frac{\alpha x}{\sqrt{\kappa}}\right) + b\sin\left(\frac{\alpha x}{\sqrt{\kappa}}\right)$$

which in general is not possible. However, we can now use the separation constant to our advantage. Recall that in Chap. 4 it was possible for any piecewise continuous function $f(x)$, $x \in [0, L]$ to be expressed as a series of trigonometric functions. In particular we can express $f(x)$ by

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin\left(\frac{\alpha_n x}{\sqrt{\kappa}}\right),$$

writing $\alpha = \alpha_n$ to emphasise its n dependence. Further, if we set

$$\frac{\alpha_n}{\sqrt{\kappa}} = \frac{n\pi}{L}, \quad n = \text{ integer}$$

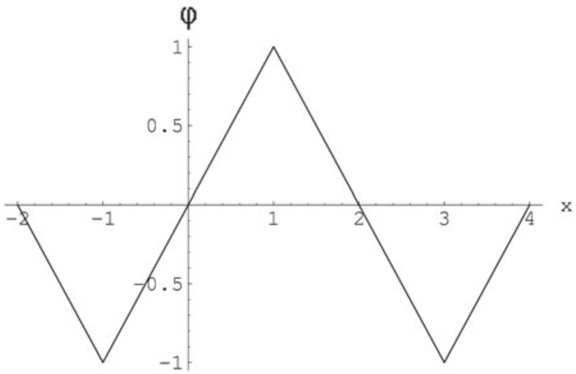
then the boundary conditions at $x = 0$ and $x = L$ are both satisfied. Here we have expressed the function $f(x)$ as a half range Fourier sine series which is consistent with the given boundary conditions. Half range cosine series or full range series can also be used of course depending on the problem.

Here, this leads to the complete solution of this particular problem in terms of the series
$$\phi(x,t) = \sum_{n=1}^{\infty} b_n e^{-(n^2\pi^2\kappa/L^2)t} \sin\left(\frac{n\pi x}{L}\right),$$

where
$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx.$$

The solution can be justified as correct using the following arguments. First of all the heat conduction equation is linear which means that we can superpose the separable solutions in the form of a series to obtain another solution provided the series is convergent. This convergence is established easily since the Fourier series itself is

Fig. 5.1 The initial distribution of $\phi(x, t)$ drawn as an odd function



pointwise convergent by construction, and the multiplying factor $e^{-(n^2\pi^2\kappa/L^2)t}$ is always less than or equal to one. The $\phi(x, t)$ obtained is thus the solution to the heat conduction equation. Here is a specific example.

Example 5.2 *Determine the solution to the boundary value problem:*

$$\begin{aligned} \frac{\partial \phi}{\partial t} &= \kappa \frac{\partial^2 \phi}{\partial x^2}, \quad x \in [0, 2] \\ \phi(x, 0) &= x + (2 - 2x)H(x - 1) \ \textit{ at time } \quad t = 0 \\ \phi(0, t) &= \phi(2, t) = 0 \ \textit{ for all time} \end{aligned}$$

where $H(x)$ is Heaviside's unit step function (see Chap. 2, Sect. 2.3*).*

Solution The form of the function $\phi(x, 0)$ is displayed in Fig. 5.1. This function is expressed as a Fourier sine series by the methods outlined in Chap. 4. This is in order to make automatic the satisfying of the boundary conditions. The Fourier sine series is not derived in detail as this belongs in Chap. 4. The result is

$$x + (2 - 2x)H(x - 1) \sim \sum_{n=1}^{\infty} \frac{8(-1)^{n-1}}{\pi^2(2n - 1)^2} \sin\left(\frac{n\pi x}{2}\right).$$

The solution to this particular boundary value problem is therefore

$$\phi(x, t) = \sum_{n=1}^{\infty} \frac{8(-1)^{n-1}}{\pi^2(2n - 1)^2} e^{-(n^2\pi^2\kappa/L^2)t} \sin\left(\frac{n\pi x}{2}\right).$$

There is much more on solving PDEs using separation of variables in specialist texts such as Zauderer (1989). We shall not dwell further on the method for its own sake, but move on to solve PDEs using Laplace transforms which itself often requires use of separation of variables.

5.4 Using Laplace Transforms to Solve PDEs

We now turn to solving the heat conduction equation using Laplace transforms. The symbol t invariably denotes time in this equation and the fact that time runs from now ($t = 0$) to infinity neatly coincides with the range of the Laplace transform. Remembering that

$$\mathcal{L}\{\phi'\} = \int_0^\infty e^{-st}\frac{d\phi}{dt}dt = s\bar{\phi}(s) - \phi(0)$$

where overbar (as usual) denotes the Laplace transform (i.e. $\mathcal{L}(\phi) = \bar{\phi}$) gives us the means of turning the PDE

$$\frac{\partial\phi}{\partial t} = \kappa\frac{\partial^2\phi}{\partial x^2}$$

into the ODE

$$s\bar{\phi}(s) - \phi(0) = \kappa\frac{d^2\bar{\phi}}{dx^2}.$$

We have of course assumed that

$$\mathcal{L}\left\{\kappa\frac{\partial^2\phi}{\partial x^2}\right\} = \int_0^\infty \kappa\frac{\partial^2\phi}{\partial x^2}e^{-st}dt = \kappa\frac{d^2}{dx^2}\int_0^\infty \phi e^{-st}dt = \kappa\frac{d^2\bar{\phi}}{dx^2}$$

which demands a continuous second order partial derivative with respect to x and an improper integral with respect to t which are well defined for all values of x so that the legality of interchanging differentiation and integration (sometimes called Leibniz' Rule) is ensured. Rather than continue in general terms, having seen how the Laplace transform can be applied to a PDE let us do an example.

Example 5.3 *Solve the heat conduction equation*

$$\frac{\partial\phi}{\partial t} = \frac{\partial^2\phi}{\partial x^2}$$

in the region $t > 0, \quad x > 0$ *with boundary conditions* $\phi(x, 0) = 0 \quad x > 0$ *(initial condition),* $\phi(0, t) = 1, \quad t > 0$ *(temperature held constant at the origin) and* $\lim_{x\to\infty}\phi(x, t) = 0$ *(the temperature a long way from the origin remains at its initial value).* [For an alternative physical interpretation of this, see the end of the solution.]

Solution Taking the Laplace transform (in t of course) gives

$$s\bar{\phi} - \phi(0) = \frac{d^2\bar{\phi}}{dx^2}$$

or

$$s\bar{\phi} = \frac{d^2\bar{\phi}}{dx^2}$$

since $\phi(x, 0) = 0$. This is an ODE with constant coefficients (not dependent on x; the presence of s does not affect this argument as there are no s derivatives) with solution

$$\bar{\phi}(x, s) = Ae^{-x\sqrt{s}} + Be^{x\sqrt{s}}.$$

Now, since $\lim_{x\to\infty} \phi(x, t) = 0$ and

$$\bar{\phi}(x, s) = \int_0^\infty \phi(x, t)e^{-st}dt$$

we have that $\bar{\phi}(x, s) \to 0$ as $x \to \infty$, hence $B = 0$. Thus

$$\bar{\phi}(x, s) = Ae^{-x\sqrt{s}}.$$

Letting $x = 0$ in the Laplace transform of ϕ gives

$$\bar{\phi}(0, s) = \int_0^\infty \phi(0, t)e^{-st}dt = \int_0^\infty e^{-st}dt = \frac{1}{s}$$

given that $\phi(0, t) = 1$ for all t. Notice that the "for all t" is crucial as the Laplace transform has integrated through all positive values of t. Inserting this boundary condition for $\bar{\phi}$ gives

$$A = \frac{1}{s}$$

whence the solution for $\bar{\phi}$ is

$$\bar{\phi}(x, s) = \frac{e^{-x\sqrt{s}}}{s}.$$

Inverting this using a table of standard forms, Appendix B or Example 3.7, gives

$$\phi(x, t) = \operatorname{erfc}\left(\frac{1}{2}xt^{-1/2}\right).$$

Another physical interpretation of this solution runs as follows. Suppose there is a viscous fluid occupying the region $x > 0$ with a flat plate on the plane $x = 0$. (Think of the plate as vertical to eliminate the effects of gravity.) The above expression for ϕ is the solution to jerking the plate in its own plane so that a velocity near the plate is generated. Viscosity takes on the role of conductivity here, the value is taken as unity in the above example. ϕ is the fluid speed dependent on time t and vertical distance from the plate x.

Most of you will realise at once that using Laplace transforms to solve this kind of PDE is not a problem apart perhaps from evaluating the inverse Laplace transform. There is a general formula for the inverse Laplace transform and we meet this informally in the next chapter and more formally in Chap. 8. You will also see that the solution

$$\phi(x,t) = \operatorname{erfc}\left(\frac{1}{2}xt^{-1/2}\right)$$

is in no way expressible in variable separable form. This particular problem is not amenable to solution using separation of variables because of the particular boundary conditions. There is a method of solution whereby we set $\xi = xt^{-1/2}$ and transform the equation $\phi_t = \phi_{xx}$ from (x,t) space into ξ space. This is called using a similarity variable. Further details can be found in more specialised texts, for example Zauderer (1989).

One more subject that is essential to consider theoretically but not discussed here is uniqueness. The heat conduction equation has a unique solution provided boundary conditions are given at $t = 0$ (initial condition) together with values of ϕ (or its derivative with respect to x) at $x = 0$ and another value ($x = a$ say) for all t. The proof of this is straightforward and makes use of contradiction. This is quite typical of uniqueness proofs, and the details can be found in, for example, Williams (1980) pp. 195–199. Although the proof of uniqueness is a side issue for a book on Laplace transforms and Fourier series, it is always important. Once a solution to a boundary value problem like the heat conduction in a bar has been found, it is vital to be sure that it is the only one. Problems that do not have a unique solution are called *ill posed* and they are not dealt with in this text. In the last twenty years, a great deal of attention has been focused on non-linear problems. These do not have a unique solution (in general) but are as far as we know accurate descriptions of real problems.

So far in this chapter we have discussed the solution of evolutionary partial differential equations, typically the heat conduction equation where initial conditions dictate the behaviour of the solution. Laplace transforms can also be used to solve second order hyperbolic equations typified by the wave equation. Let us see how this is done. The one-dimensional wave equation may be written

$$\frac{\partial^2\phi}{\partial t^2} = c^2\frac{\partial^2\phi}{\partial x^2}$$

where c is a constant called the celerity or wave speed, x is the displacement and t of course is time. The kind of problem that can be solved is that for which conditions at time $t = 0$ are specified. This does not cover all wave problems and may not even be considered to be typical of wave problems, but although problems for which conditions at two different times are given can be solved using Laplace transforms (see Chap. 3) alternative solution methods (e.g. using characteristics) are usually better. As before, the technique is to take the Laplace transform of the equation with respect to time, noting that this time there is a second derivative to deal with. Defining

$$\bar{\phi} = \mathcal{L}\{\phi\}$$

and using

$$\mathcal{L}\left\{\frac{\partial^2 \phi}{\partial t^2}\right\} = s^2\bar{\phi} - s\phi(0) - \phi'(0)$$

the wave equation transforms into the following ordinary differential equation for $\bar{\phi}$:-

$$s^2\bar{\phi} - s\phi(0) - \phi'(0) = c^2\frac{d^2\bar{\phi}}{dx^2}.$$

The solution to this equation is conveniently written in terms of hyperbolic functions since the x boundary conditions are almost always prescribed at two finite values of x. If the problem is in an infinite half space, for example the vibration of a very long beam fixed at one end (a cantilever), the complementary function $Ae^{sx/c} + Be^{-sx/c}$ can be useful. As with the heat conduction equation, rather than pursuing the general problem, let us solve a specific example.

Example 5.4 *A beam of length a initially at rest has one end fixed at $x = 0$ with the other end free to move at $x = a$. Assuming that the beam only moves longitudinally (in the x direction) and is subject to a constant force ED along its length where E is the Young's modulus of the beam and D is the displacement per unit length, find the longitudinal displacement of the beam at any subsequent time. Find also the motion of the free end at $x = a$.*

Solution The first part of this problem has been done already in that the Laplace transform of the one dimensional wave equation is

$$s^2\bar{\phi} - s\phi(0) - \phi'(0) = c^2\frac{d^2\bar{\phi}}{dx^2}$$

where $\phi(x, t)$ is now the displacement, $\bar{\phi}$ its Laplace transform and we have assumed that the beam is perfectly elastic, hence actually obeying the one dimensional wave equation. The additional information we have available is that the beam is initially at rest, $x = 0$ is fixed for all time and that the other end at $x = a$ is free. These are translated into mathematics as follows:-

$$\phi(x, 0) = 0, \quad \frac{\partial \phi}{\partial t}(x, 0) = 0 \text{ (beam is initially at rest)}$$

together with

$$\phi(0, t) = 0 \ (x = 0 \text{ fixed for all time})$$

and

$$\frac{\partial \phi(a, t)}{\partial x} = D \ \ \forall t \text{ (the end at } x = a \text{ is free).}$$

Hence

$$\frac{d^2\bar{\phi}}{dx^2} = \frac{s^2}{c^2}\bar{\phi}$$

the solution of which can be written

$$\bar{\phi}(x, s) = k_1 \cosh\left(\frac{sx}{c}\right) + k_2 \sinh\left(\frac{sx}{c}\right).$$

In order to find k_1 and k_2, we use the x boundary condition. If $\phi(0, t) = 0$ then, taking the Laplace transform, $\bar{\phi}(0, s) = 0$ so $k_1 = 0$. Hence

$$\bar{\phi}(x, s) = k_2 \sinh\left(\frac{sx}{c}\right) \text{ and } \frac{d\bar{\phi}}{dx} = \frac{sk_2}{c} \cosh\left(\frac{sx}{c}\right).$$

We have that

$$\frac{\partial \phi}{\partial x} = D \text{ at } x = a \quad \text{for all } t.$$

The Laplace transform of this is

$$\frac{d\bar{\phi}}{dx} = \frac{D}{s} \text{ at } x = a.$$

Hence

$$\frac{D}{s} = \frac{sk_2}{c} \cosh\left(\frac{sa}{c}\right)$$

from which

$$k_2 = \frac{Dc}{s^2 \cosh(\frac{sa}{c})}.$$

Hence the solution to the ODE for $\bar{\phi}$ is

$$\bar{\phi} = \frac{Dc \sinh\left(\frac{sx}{c}\right)}{s^2 \cosh\left(\frac{sa}{c}\right)}$$

and is completely determined.

The remaining problem, and it is a significant one, is that of inverting this Laplace transform to obtain the solution $\phi(x, t)$. Many would use the inversion formula of Chap. 8, but as this is not available to us, we scan the table of standard Laplace transforms (Appendix B) and find the result

$$\mathcal{L}^{-1}\left\{\frac{\sinh(sx)}{s^2 \cosh(sa)}\right\} = x + \frac{8a}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)^2}\sin\left(\frac{(2n-1)\pi x}{2a}\right)\cos\left(\frac{(2n-1)\pi t}{2a}\right)$$

and deduce that

$$\phi(x,t) = Dx + \frac{8aD}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)^2}\sin\left(\frac{(2n-1)\pi x}{2a}\right)\cos\left(\frac{(2n-1)\pi t}{2a}\right).$$

It is not until we meet the general inversion formula in Chap. 8 that we are able to see how such formulae are derived using complex analysis.

Example 5.5 *Starting with the linear Rossby wave equation*

$$\nabla^2\psi_t + \beta\psi_x = 0$$

determine the long term response of an initially quiescent ocean if the variations in the y direction are much smaller than those in the x direction.

Solution The linear Rossby wave equation takes the form

$$\nabla^2\psi_t + \beta\psi_x = 0.$$

Take Laplace transforms (in t) using:

$$\mathcal{L}\{\psi\} = \bar{\psi} \quad \mathcal{L}\{\psi_t\} = s\bar{\psi} - \psi(x,0)$$

giving:

$$s\nabla^2\bar{\psi} + \beta\bar{\psi}_x = \nabla^2\psi(x,0).$$

Initially we can assume that the streamfunction $\psi(x,0) = \psi_0$ a constant. Thus the Rossby wave equation becomes:

$$s\left(\frac{\partial^2\bar{\psi}}{\partial x^2} + \frac{\partial^2\bar{\psi}}{\partial y^2}\right) + \beta\frac{\partial\bar{\psi}}{\partial x} = 0.$$

In order to solve this we try a solution of the form:

$$\bar{\psi} = q(x)e^{iky} \quad \text{so that} \quad \frac{\partial^2\bar{\psi}}{\partial y^2} = -k^2\bar{\psi}$$

and the Rossby wave partial differential equation leads to the following ordinary differential equation for q.

$$sq'' + \beta q' - sk^2 q = 0.$$

This is an equation with constant coefficients (s is considered a constant as it is not the object of differentiation in this problem). Thus we take the normal route of assuming a solution in the form $q = \exp(\lambda x)$ to get a quadratic for λ which is:

$$s\lambda^2 + \beta\lambda - sk^2 = 0$$

whence

$$\lambda = \frac{-\beta \pm \sqrt{\beta^2 + 4s^2k^2}}{2s}.$$

This entire problem arises from considering the onset of the monsoon over an infinite Indian Ocean, hence the initial response of the ocean is less important than the response for large times. Thus we expand these two solutions for λ in power series in s and consider small values of s (corresponding to large time). Hence

$$q \sim e^{-\beta x/s} \quad \text{and} \quad q \sim e^{-sxk^2/\beta}.$$

The simplest case is to consider no variation in y so set $k = 0$. The boundary condition $\psi = \psi_0$ (a constant) transforms to $\bar{\psi} = \psi_0/s$. Thus

$$\bar{\psi}(x, s) = \frac{1}{s}e^{-\beta x/s}.$$

We have the standard form, see Appendix B:

$$\mathcal{L}^{-1}\left(\frac{e^{-a/s}}{s^{n+1}}\right) = \left(\frac{t}{a}\right)^{n/2} J_n(2\sqrt{at})$$

where J_n is the Bessel function of the first kind of integer order n, so put $a = \beta x$ and $n = 0$ and take inverse Laplace transforms to obtain

$$\psi(x, t) = \psi_0 J_0(2\sqrt{\beta x t}).$$

This is the basic solution. Changing the time origin from 0 to t_0 and recognising that $\psi = \psi_0$ is another trivial solution leads to the "well known" oceanographic solution to the problem of modelling the onset of the Somali current:

$$\psi(x, t) = \psi_0[1 - J_0(2\sqrt{\beta x(t - t_0)})].$$

So that at time $t = t_0$ the streamfunction ψ is a constant but after the wind has blown for a while, it is given by the above function. The solution represents a northerly flow that gradually narrows. This is shown from the relationship between streamfunction and v the northerly flowing current.

5.5 Boundary Conditions and Asymptotics

A partial differential equation together with enough boundary conditions to ensure the existence of a unique solution is called a boundary value problem, sometimes abbreviated to BVP. Parabolic and hyperbolic equations of the type suited to solution using Laplace transforms are defined over a semi infinite domain. In two dimensional Cartesian co-ordinates, this domain will take the form of a semi infinite strip shown in Fig. 5.2. Since one of the variables is almost always time, the problem has conditions given at time $t = 0$ and the solution once found is valid for all subsequent times, theoretically to time $t = \infty$. Indeed, this is the main reason why Laplace transforms are so useful for these types of problems. Thus, in transformed space, the boundary condition at $t = 0$ and the behaviour of the solution as $t \to \infty$ get swept up in the Laplace transform and appear explicitly in the transformed equations, whereas the conditions on the other variable (x say) which form the two infinite sides of the rectangular domain of Fig. 5.2 get transformed into two boundary conditions that are the end points of a two point boundary value problem (one dimensional) in x. The Laplace transform variable s becomes passive because no derivatives with respect to s are present. If there are three variables (or perhaps more) and one of them is time-like in that conditions are prescribed at $t = 0$ and the domain extends for arbitrarily large time, then similar arguments prevail. However, the situation is a little more complicated in that the transformed boundary value problem has only had its dimension reduced by one. That is, a three dimensional heat conduction equation which takes the form

$$\frac{\partial \phi}{\partial t} = \kappa \nabla^2 \phi$$

where

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}$$

which is in effect a four dimensional equation involving time together with three space dimensions, is transformed into a Poisson like equation:-

$$\kappa \nabla^2 \bar{\phi}(x, y, z, s) = s\bar{\phi}(x, y, z, s) - \phi(x, y, z, 0)$$

where, as usual, $\mathcal{L}\{\phi(x, y, z, t)\} = \bar{\phi}(x, y, z, s)$.

It remains difficult in general to determine the inverse Laplace transform, so various properties of the Laplace transform are invoked as an alternative to complete inversion. One device is particularly useful and amounts to using a special case of Watson's lemma, a well known result in asymptotic analysis. If for small values of t, $\phi(x, y, z, t)$ has a power series expansion of the form

$$\phi(x, y, z, t) = \sum_{n=0}^{\infty} a_n(x, y, z) t^{k+n}$$

and $|\phi|e^{-ct}$ is bounded for some k and c, then the result of Chap. 2 following the Final Value Theorem (Theorem 2.7) can be invoked and we can deduce that as $s \to \infty$ the Laplace transform of ϕ, $\bar{\phi}$ has an equivalent asymptotic expansion

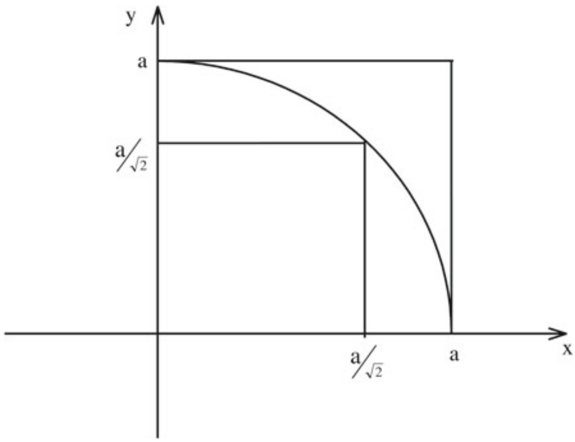


Fig. 5.2 The domain

$$\bar{\phi}(x, y, z, s) = \sum_{n=0}^{\infty} a_n(x, y, z) \frac{\Gamma(n + k + 1)}{s^{n+k+1}}.$$

Now asymptotic expansions are not necessarily convergent series; however the first few terms do give a good approximation to the function $\bar{\phi}(x, y, z, s)$. What we have here is an approximation to the transform of ϕ that relates the form for large s to the behaviour of the original variable ϕ for small t. Note that this is consistent with the initial value theorem (Theorem 2.6). It is sometimes the case that each series is absolutely convergent. A term by term evaluation can then be justified. However, often the most interesting and useful applications of asymptotic analysis take place when the series are not convergent. The classical text by Copson (1967) remains definitive. The serious use of these results demands a working knowledge of complex variable theory, in particular of poles of complex functions and residue theory. These are not dealt with until Chap. 8 so examples involving complex variables are postponed until then. Here is a reasonably straightforward example using asymptotic series, just to get the idea.

Example 5.6 *Find an approximation to the solution of the partial differential equation*

$$\frac{\partial \phi}{\partial t} = c^2 \frac{\partial^2 \phi}{\partial x^2}$$

for small times where $\phi(x, 0) = \cos(x)$, *by using an asymptotic series.*

Solution It is possible to solve this BVP exactly, but let us take Laplace transforms to obtain

$$s\bar{\phi} - \cos(x) = c^2 \frac{d^2 \bar{\phi}}{dx^2}$$

then try an asymptotic series of the form

$$\bar{\phi}(x,s) = \sum_{n=0}^{\infty} \frac{b_n(x)}{s^{n+k+1}}$$

valid far enough away from the singularities of $\bar{\phi}$ in s space. (This is only true provided $\bar{\phi}$ does not have branch points. See Sect. 8.4 for more about branch points.) Equating coefficients of $1/s^n$ yields straight away that $k = 0$, then we get

$$b_0 = \cos(x); \quad b_1 = c^2 \frac{d^2 b_0}{dx^2}; \quad b_2 = c^2 \frac{d^2 b_1}{dx^2}; \ldots$$

and hence

$$b_1 = -c^2 \cos(x), \quad b_2 = c^4 \cos(x), \quad b_3 = -c^6 \cos(x) \;\; \text{etc.}$$

This yields

$$\bar{\phi}(x,s) = \cos(x) \sum_{n=0}^{\infty} \frac{(-1)^n c^{2n}}{s^{n+2}}.$$

Provided we have $s > c > 0$, term by term inversion is allowable here as the series will then converge for all values of x. It is uniformly but not absolutely convergent. This results in

$$\phi(x,t) = \cos(x) \sum_{n=0}^{\infty} \frac{(-1)^n c^{2n} t^n}{n!}$$

which is immediately recognised as

$$\phi(x,t) = \cos(x) e^{-c^2 t}$$

a solution that could have been obtained directly using separation of variables. As is obvious from this last example, we are hampered in the kind of problem that can be solved because we have yet to gain experience of the use of complex variables. Fourier transforms are also a handy tool for solving certain types of BVP, and these are the subject of the next chapter. So finally in this chapter, a word about other methods of solving partial differential equations.

In the years since the development of the workstation and desk top microcomputer, there has been a revolution in the methods used by engineers and applied scientists in industry who need to solve boundary value problems. In real situations, these problems are governed by equations that are far more complicated than those covered here. Analytical methods can only give a first approximation to the solution of such problems and these methods have been surpassed by numerical methods based on finite difference and finite element approximations to the partial differential equations. However, it is still essential to retain knowledge of analytical methods, as these give insight as to the general behaviour in a way that a numerical solution will never do, and because analytical methods actually can lead to an increase in efficiency in the numerical method eventually employed to solve the real problem. For example, an analytical method may be able to tell where a solution changes rapidly even though the solution itself cannot be obtained in closed form and this helps to design the numerical procedure, perhaps even suggesting alternative methods (adaptive gridding) but more likely helping to decide on co-ordinate systems and step lengths. The role of mathematics has thus changed in emphasis from providing direct solutions to real problems to giving insight into the underlying properties of the solutions. As far as this chapter is concerned, future applications of Laplace transforms to partial differential equations are met briefly in Chap. 8, but much of the applications are too advanced for this book and belong in specialist texts on the subject, e.g. Weinberger (1965).

5.6 Exercises

1. Using separation of variables, solve the boundary value problem:

$$\frac{\partial\phi}{\partial t} = \kappa\frac{\partial^2\phi}{\partial x^2}, \quad x \in \left[0, \frac{\pi}{4}\right]$$
$$\phi(x, 0) = x\left(\frac{\pi}{4} - x\right) \text{ at time } \quad t = 0$$
$$\phi(0, t) = \phi\left(\frac{\pi}{4}, t\right) = 0 \text{ for all time,}$$

using the methods of Chap. 4 to determine the Fourier series representation of the function $x\left(\frac{\pi}{4} - x\right)$.

2. The function $\phi(x, t)$ satisfies the PDE

$$a\frac{\partial^2\phi}{\partial x^2} - b\frac{\partial\phi}{\partial x} - \frac{\partial\phi}{\partial t} = 0$$

with $x > 0$, $a > 0$, $b > 0$ and boundary conditions $\phi(x, 0) = 0$ for all x, $\phi(0, t) = 1$ for all t and $\phi \to 0$ as $x \to \infty$, $t > 0$. Use Laplace transforms to find $\bar{\phi}$, the Laplace transform of ϕ. (Do not attempt to invert it.)

3. Use Laplace transforms to solve again the BVP of Exercise 1 but this time in the form

$$\phi(x, t) = -x^2 + \frac{\pi}{4}x - 2\kappa t + 2\kappa\mathcal{L}^{-1}\left\{\frac{\sinh(\frac{\pi}{4} - x)\sqrt{\frac{s}{\kappa}}}{s^2\sinh(\frac{\pi}{4})\sqrt{\frac{s}{\kappa}}}\right\}$$
$$+ 2\kappa\mathcal{L}^{-1}\left\{\frac{\sinh(x\sqrt{\frac{s}{\kappa}})}{s^2\sinh(\frac{\pi}{4}\sqrt{\frac{s}{\kappa}})}\right\}.$$

Use the table of Laplace transforms to invert this expression. Explain any differences between this solution and the answer to Exercise 1.

4. Solve the PDE
$$\frac{\partial^2 \phi}{\partial x^2} = \frac{\partial \phi}{\partial y}$$
with boundary conditions $\phi(x, 0) = 0, \quad \phi(0, y) = 1, \quad y > 0$ and
$$\lim_{x \to \infty} \phi(x, y) = 0.$$

5. Suppose that $u(x, t)$ satisfies the equation of telegraphy
$$\frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} - \frac{k}{c^2} \frac{\partial u}{\partial t} + \frac{1}{4} \frac{k^2}{c^2} u = \frac{\partial^2 u}{\partial x^2}.$$
Find the equation satisfied by $\phi = u e^{-kt/2}$, and hence use Laplace transforms (in t) to determine the solution for which
$$u(x, 0) = \cos(mx), \quad \frac{\partial u}{\partial t}(x, 0) = 0 \text{ and } u(0, t) = e^{kt/2}.$$

6. The function $u(x, t)$ satisfies the BVP
$$u_t - c^2 u_{xx} = 0, \quad x > 0, \quad t > 0, \quad u(0, t) = f(t), \quad u(x, 0) = 0$$
where $f(t)$ is piecewise continuous and of exponential order. (The suffix derivative notation has been used.) Find the solution of this BVP by using Laplace transforms together with the convolution theorem. Determine the explicit solution in the special case where $f(t) = \delta(t)$, where $\delta(t)$ is the Dirac δ function.

7. A semi-infinite solid occupying the region $x > 0$ has its initial temperature set to zero. A constant heat flux is applied to the face at $x = 0$, so that $T_x(0, t) = -\alpha$ where T is the temperature field and α is a constant. Assuming linear heat conduction, find the temperature at any point x ($x > 0$) of the bar and show that the temperature at the face at time t is given by
$$\alpha \sqrt{\frac{\kappa}{\pi t}}$$
where κ is the thermal conductivity of the bar.

8. Use asymptotic series to provide an approximate solution to the wave equation
$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$
valid for small values of t with

$$u(x, 0) = 0, \quad \frac{\partial u}{\partial t}(x, 0) = \cos(x).$$

9. Repeat the last exercise, but using instead the boundary conditions

$$u(x, 0) = \cos(x), \quad \frac{\partial u}{\partial t}(x, 0) = 0.$$

Chapter 6
Fourier Transforms

6.1 Introduction

It is not until a little later in this chapter that we define the Fourier transform; it is appropriate to arrive at it through the mathematics of the previous chapters. There are two ways of approaching the subject of Fourier transforms, both ways are open to us. One way is to carry on directly from Chap. 4 and define Fourier transforms in terms of the mathematics of linear spaces by carefully increasing the period of the function $f(x)$. This would lead to the Fourier series we defined in Chap. 4 becoming, in the limit of infinite period, an integral. This integral leads directly to the Fourier transform. On the other hand, the Fourier transform can be straightforwardly defined as an example of an integral transform and its properties compared and in many cases contrasted with those of the Laplace transform. It is this second approach that is favoured here, with the first more pure mathematical approach outlined towards the end of Sect. 6.2. This choice is arbitrary, but it is felt that the more "hands on" approach should dominate here. Having said this, texts that concentrate on computational aspects such as the FFT (Fast Fourier Transform), on time series analysis and on other branches of applied statistics sometimes do prefer the more pure approach in order to emphasise precision. Also, there is in the next chapter an introduction to wavelets. Wavelets are particularly suited to the analysis of time series and so this gives us another reason for us to favour the second approach here and leave the relation between wavelets and Fourier series to the next chapter.

6.2 Deriving the Fourier Transform

Definition 6.1 *Let f be a function defined for all $x \in \mathbb{R}$ with values in $\mathbb{C}$. The Fourier transform is a mapping $F : \mathbb{R} \to \mathbb{C}$ defined by*

$$F(\omega) = \int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx.$$

P. Dyke, *An Introduction to Laplace Transforms and Fourier Series*,
Springer Undergraduate Mathematics Series, DOI: 10.1007/978-1-4471-6395-4_6,

Of course, for some $f(x)$ the integral on the right does not exist. We shall spend some time discussing this a little later. There can be what amounts to trivial differences between definitions involving factors of 2π or $\sqrt{2\pi}$. Although this is of little consequence mathematically, it is important to stick to the definition whichever version is chosen. In engineering or medicine where x is often time, and ω frequency, factors of 2π or $\sqrt{2\pi}$ can make a lot of difference.

If $F(\omega)$ is defined by the integral above, then it can be shown that

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{i\omega x}d\omega.$$

This is the inverse Fourier transform. It is instructive to consider $F(\omega)$ as a complex valued function of the form

$$F(\omega) = A(\omega)e^{i\phi(\omega)}$$

where $A(\omega)$ and $\phi(\omega)$ are real functions of the real variable ω. F is thus a complex valued function of a real variable ω. Some readers will recognise $F(\omega)$ as a spectrum function, hence the letters A and ϕ which represent the amplitude and phase of F respectively. We shall not dwell on this here however. If we merely substitute for $F(\omega)$ we obtain

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} A(\omega)e^{i\omega x + i\phi(\omega)}d\omega.$$

We shall return to this later when discussing the relationship between Fourier transforms and Fourier series. Let us now consider what functions permit Fourier transforms. A glance at the definition tells us that we cannot for example calculate the Fourier transform of polynomials or even constants due to the oscillatory nature of the kernel. This is a feature that might seem to render the Fourier transform useless. It is certainly a difficulty, but one that is more or less completely solved by extending what is meant by an integrable function through the use of generalised functions. These were introduced in Sect. 2.6, and it turns out that the Fourier transform of a constant is closely related to the Dirac δ function defined in Sect. 2.6. The impulse function is a representative of this class of functions and we met many of its properties in Chap. 2. In that chapter, mention was also made of the use of the impulse function in many applications, especially in electrical engineering and signal processing. The general mathematics of generalised functions is outside the scope of this text, but more of its properties will be met later in this chapter.

If we write the function to be transformed in the form $e^{-kx}f(x)$ then the Fourier transform is the integral

$$\int_{-\infty}^{\infty} e^{-i\omega x}e^{-kx}f(x)dx$$

straight from the definition. In this form, the Fourier transform can be related to the Laplace transform. First of all, write

$$F_k(\omega) = \int_0^\infty e^{-(k+i\omega)x} f(x)dx$$

then $F_k(\omega)$ will exist provided the function $f(x)$ is of exponential order (see Chap. 1). Note too that the bottom limit has become 0. This reflects that the variable x is usually time. The inverse of $F_k(\omega)$ is straightforward to find once it is realised that the function $f(x)$ can be defined as identically zero for $x < 0$. Whence we have

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega x} F_k(\omega)d\omega = \begin{cases} 0 & x < 0 \\ e^{-kx} f(x) & x \geq 0. \end{cases}$$

An alternative way of expressing this inverse is in the form

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{(k+i\omega)x} F_k(\omega)d\omega = \begin{cases} 0 & x < 0 \\ f(x) & x \geq 0. \end{cases}$$

In this formula, and in the one above for $F_k(\omega)$, the complex number $k + i\omega$ occurs naturally. This is a variable, and therefore a *complex* variable and it corresponds to s the Laplace transform variable defined in Chap. 1. Now, the integral on the left of the last equation is not meaningful if we are to regard $k + i\omega = s$ as the variable. As k is not varying, we can simply write

$$ds = id\omega$$

and s will take the values $k - i\infty$ and $k + i\infty$ at the limits $\omega = -\infty$ and $\omega = \infty$ respectively. The left-hand integral is now, when written as an integral in s,

$$\frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty} e^{sx} F(s)ds$$

where $F(s) = F_k(\omega)$ is now a complex valued function of the complex variable s. Although there is nothing illegal in the way the variable changes have been undertaken in the above manipulations, it does amount to a rather cartoon derivation. A more rigorous derivation of this integral is given in Chap. 8 after complex variables have been properly introduced. The formula

$$f(x) = \frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty} e^{sx} F(s)ds$$

is indeed the general form of the inverse Laplace transform, given $F(s) = \mathcal{L}\{f(x)\}$.

We now approach the definition of Fourier transforms from a different viewpoint. In Chap. 4, Fourier series were discussed at some length. As a summary for present purposes, if $f(x)$ is a periodic function, and for simplicity let us take the period as being 2π (otherwise in all that follows replace x by $lx/2\pi$ where l is the period) then $f(x)$ can be expressed as the Fourier series

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx))$$

where

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) dx, \quad n = 0, 1, 2, \ldots$$

and

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) dx \quad n = 0, 1, 2, \ldots.$$

These have been derived in Chap. 4 and follow from the orthogonality properties of sine and cosine. The factor $\frac{1}{2}$ in the constant term enables a_n to be expressed as the integral shown without $n = 0$ being an exception. It is merely a convenience, and as with factors of 2π in the definition of Fourier transform, the definition of a Fourier series can have these trivial differences. The task now is to see how Fourier series can be used to generate not only the Fourier transform but also its inverse. The first step is to convert sine and cosine into exponential form; this will re-derive the complex form of the Fourier series first done in Chap. 4. Such a re-derivation is necessary because of the slight change in definition of Fourier series involving the a_0 term. So we start with the standard formulae

$$\cos(nx) = \frac{1}{2}(e^{inx} + e^{-inx})$$

and

$$\sin(nx) = \frac{1}{2i}(e^{inx} - e^{-inx}).$$

Some algebra of an elementary nature is required before the Fourier series as given above is converted into the complex form

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

where

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx.$$

The complex numbers c_n are related to the real numbers a_n and b_n by the simple formulae

$$c_n = \frac{1}{2}(a_n - ib_n), \quad c_{-n} = \overline{c_n}, \quad n = 0, 1, 2, \ldots$$

and it is assumed that $b_0 = 0$. The overbar denotes complex conjugate. There are several methods that enable one to move from this statement of complex Fourier

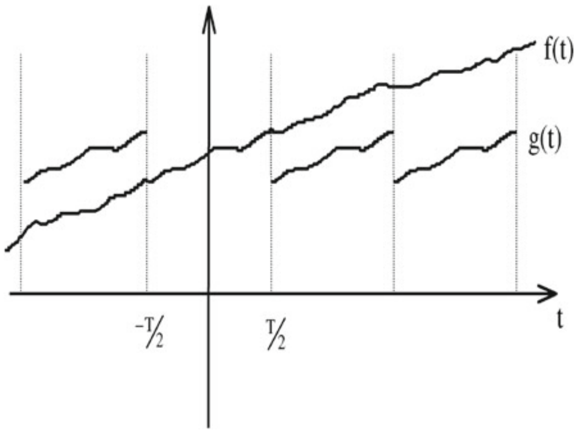


Fig. 6.1 The function $f(t)$ and its periodic clone $g(t)$

series to Fourier transforms. The method adopted here is hopefully easy to follow as it is essentially visual. First of all consider a function $g(t)$ which has period T. (We have converted from x to t as the period is T and no longer 2π. It is the transformation $x \to 2\pi t/T$). Figure 6.1 gives some insight into what we are trying to do here. The functions $f(t)$ and $g(t)$ coincide precisely in the interval $[-\frac{T}{2}, \frac{T}{2}]$, but not necessarily outside this range. Algebraically we can write

$$g(t) = \begin{cases} f(t) & |t| < \frac{1}{2}T \\ \\ f(t-nT) & \frac{1}{2}(2n-1)T < |t| < \frac{1}{2}(2n+1)T \end{cases}$$

where n is an integer. Since $g(t)$ is periodic, it possesses a Fourier series which, using the complex form just derived, can be written

$$g(t) = \sum_{n=-\infty}^{\infty} G_n e^{in\omega_0 t}$$

where G_n is given by

$$G_n = \frac{1}{T}\int_{-T/2}^{T/2} g(t)e^{-in\omega_0 t}dt$$

and $\omega_0 = 2\pi/T$ is the frequency. Again, this is obtained straightforwardly from the previous results in x by writing

$$x = \frac{2\pi t}{T} = \omega_0 t.$$

We now combine these results to give

$$g(t) = \sum_{n=-\infty}^{\infty} \left[\frac{1}{T} \int_{-T/2}^{T/2} g(t) e^{-in\omega_0 t} dt \right] e^{in\omega_0 t}.$$

The next step is the important one. Note that

$$n\omega_0 = \frac{2\pi n}{T}$$

and that the difference in frequency between successive terms is ω_0. As we need this to get smaller and smaller, let $\omega_0 = \Delta\omega$ and $n\omega_0 = \omega_n$ which is to remain finite as $n \to \infty$ and $\omega_0 \to 0$ together. The integral for G_n can thus be re-written

$$G = \int_{-T/2}^{T/2} g(t) e^{-i\omega_n t} dt.$$

Having set everything up, we are now in a position to let $T \to \infty$, the mathematical equivalent of lighting the blue touchpaper. Looking at Fig. 6.1 this means that the functions $f(t)$ and $g(t)$ coincide, and

$$\begin{aligned} g(t) &= \lim_{T\to\infty} \sum_{n=-\infty}^{\infty} G e^{i\omega_n t} \frac{\Delta\omega}{2\pi} \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} G e^{i\omega t} d\omega \end{aligned}$$

with

$$G(\omega) = \int_{-\infty}^{\infty} g(t) e^{-i\omega t} dt.$$

We have let $T \to \infty$, replaced $\Delta\omega$ by the differential $d\omega$ and ω_n by the variable ω. All this certainly lies within the definition of the improper Riemann integral given in Chap. 1. We thus have

$$G(\omega) = \int_{-\infty}^{\infty} g(t) e^{-i\omega t} dt$$

with

$$g(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(\omega) e^{i\omega t} d\omega.$$

This coincides precisely with the definition of Fourier transform given at the beginning of this chapter, right down to where the factor of 2π occurs. As has already been said, this positioning of the factor 2π is somewhat arbitrary, but it important to be consistent, and where it is here gives the most convenient progression to Laplace transforms as indicated earlier in this section and in Chap. 8.

In getting the Fourier transform pair (as g and G are called) we have lifted the restriction that $g(t)$ be a periodic function. We have done this at a price however in that the improper Riemann integrals must be convergent. As we have already stated, unfortunately this is not the case for a wide class of functions including elementary functions such as sine, cosine, and even the constant function. However this serious problem is overcome through the development of generalised functions such as Dirac's δ function (see Chap. 2) and also through wavelets that are introduced in the next chapter.

6.3 Basic Properties of the Fourier Transform

There are as many properties of the Fourier transform as there are of the Laplace transform. These involve shift theorems, transforming derivatives, etc. but they are not so widely used simply due to the restrictions on the class of functions that can be transformed. Most of the applications lie in the fields of electrical and electronic engineering which are full of the jumpy and impulse like functions to which Fourier transforms are particularly suited. Here is a simple and quite typical example.

Example 6.1 *Calculate the Fourier transform of the "top hat" or rectangular pulse function defined as follows:-*

$$f(t) = \begin{cases} A & |t| \le T \\ 0 & |t| > T \end{cases}$$

where A is a constant (amplitude of the pulse) and T is a second constant (width of the pulse).

Solution Evaluation of the integral is quite straightforward and the details are as follows

$$\begin{aligned} F(\omega) &= \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt \\ &= \int_{-T}^{T} Ae^{-i\omega t}dt \\ &= \left[-\frac{A}{i\omega}e^{-i\omega t}\right]_{-T}^{T} \\ F(\omega) &= \frac{2A}{\omega}\sin(\omega T). \end{aligned}$$

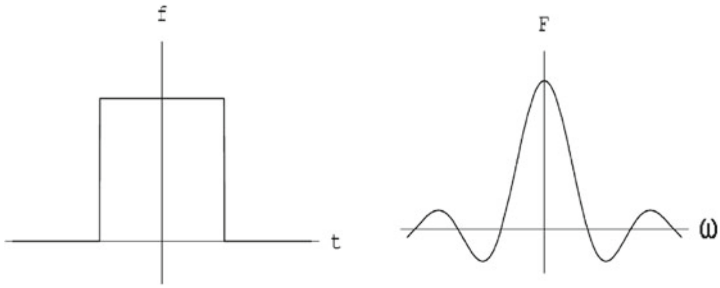


Fig. 6.2 The square wave function $f(t)$, and its Fourier transform $F(\omega)$

Mathematically this is routine and rather uninteresting. However the graphs of $f(t)$ and $F(\omega)$ are displayed side by side in Fig. 6.2, and it is worth a little discussion.

The relationship between $f(t)$ and $F(\omega)$ is that between a function of time ($f(t)$) and the frequencies that this function (called a signal by engineers) contains, $F(\omega)$. The subject of spectral analysis is a large one and sections of it are devoted to the relationship between a spectrum (often called a power spectrum) of a signal and the signal itself. This subject has been particularly fast growing since the days of the first satellite launch and the advent of satellite, then cable, and now digital television ensures its continuing growth. During much of the remainder of this chapter this kind of application will be hinted at, but a full account is of course not possible. The complex nature of $F(\omega)$ is not a problem. Most time series are not symmetric, so the modulus of $F(\omega)$ ($A(\omega)$) carries the frequency information.

A more typical-looking signal is shown on the left in Fig. 6.3. Signals do not have a known functional form, and so their Fourier transforms cannot be determined in closed form either. However some general characteristics are depicted on the right hand side of this figure. Only the modulus can be drawn in this form as the Fourier transform is in general a complex quantity. The kind of shape $|F(\omega)|$ has is also fairly typical. High frequencies are absent as this would imply a rapidly oscillating signal; similarly very low frequencies are also absent as this would imply that the signal very rarely crossed the t axis. Thus the graph of $|F(\omega)|$ lies entirely between $\omega = 0$ and a finite value. Of course, any positive variation is theoretically possible between these limits, but the single maximum is most common. There is a little more to be said about these ideas in Sect. 6.5 when windowing is discussed, and in the next chapter.

Sometimes it is inconvenient to deal with explicitly complex quantities, and the Fourier transform is expressed in real and imaginary form as follows. If

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt$$

then

$$F_c(\omega) = \int_{0}^{\infty} f(t)\cos(\omega t)dt$$

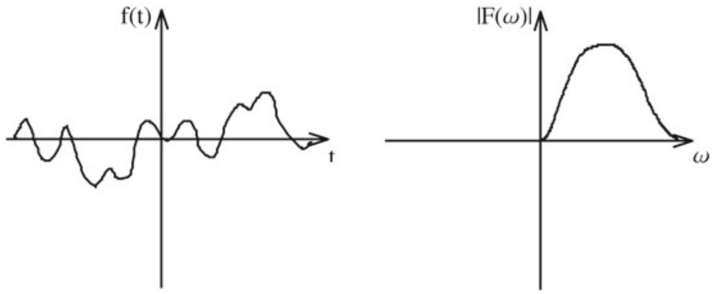


Fig. 6.3 A typical wave form $f(t)$ and the amplitude of its Fourier transform $|F(\omega)| = A(\omega)$

is the Fourier cosine transform, and

$$F_s(\omega) = \int_0^\infty f(t)\sin(\omega t)dt$$

is the Fourier sine transform. We note that the bottom limits in both the Fourier cosine and sine transform are zero rather than $-\infty$. This is in keeping with the notion that in practical applications t corresponds to time. Once more we warn that differences from these definitions involving positioning of the factor π are not uncommon. From the above definition it is easily deduced that

$$F(\omega) = \int_0^\infty [f(t) + f(-t)]\cos(\omega t)dt - i\int_0^\infty [f(t) - f(-t)]\sin(\omega t)dt$$

so if f is an odd function $[f(t) = -f(-t)]$, $F(\omega)$ is pure imaginary, and if f is an even function $[f(t) = f(-t)]$, $F(\omega)$ is real. We also note that if the bottom limit on each of the Fourier sine and cosine transforms remained at $-\infty$ as in some texts, then the Fourier sine transform of an even function is zero as is the Fourier cosine transform of an odd function. This gives another good reason for the zero bottom limits for these transforms. Now let us examine some of the more common properties of Fourier transforms, starting with the inverses of the sine and cosine transforms. These are unsurprising: if

$$F_c(\omega) = \int_0^\infty f(t)\cos(\omega t)dt$$

then

$$f(t) = \frac{2}{\pi}\int_0^\infty F_c(\omega)\cos(\omega t)d\omega$$

and if

$$F_s(\omega) = \int_0^\infty f(t)\sin(\omega t)dt$$

then

$$f(t) = \frac{2}{\pi}\int_0^\infty F_s(\omega)\sin(\omega t)d\omega.$$

The proof of these is left as an exercise for the reader. The first property of these transforms we shall examine is their ability to evaluate certain improper real integrals in closed form. Most of these integrals are challenging to evaluate by other means (although readers familiar with the residue calculus also found in summary form in Chap. 8 should be able to do them). The following example illustrates this.

Example 6.2 *By considering the Fourier cosine and sine transforms of the function* $f(t) = e^{-at}$, *a a constant, evaluate the two integrals*

$$\int_0^\infty \frac{\cos(kx)}{a^2+x^2}\,dx \text{ and } \int_0^\infty \frac{x\sin(kx)}{a^2+x^2}\,dx.$$

Solution First of all note that the cosine and sine transforms can be conveniently combined to give

$$\begin{aligned} F_c(\omega) + iF_s(\omega) &= \int_0^\infty e^{(-a+i\omega)t}dt \\ &= \left[\frac{1}{-a+i\omega}e^{(-a+i\omega)t}\right]_0^\infty \\ &= \frac{1}{a-i\omega} = \frac{a+i\omega}{a^2+\omega^2} \end{aligned}$$

whence

$$F_c(\omega) = \frac{a}{a^2+\omega^2} \text{ and } F_s(\omega) = \frac{\omega}{a^2+\omega^2}.$$

Using the formula given for the inverse transforms gives

$$\frac{2}{\pi}\int_0^\infty \frac{a}{a^2+\omega^2}\cos(\omega t)d\omega = e^{-at}$$

and

$$\frac{2}{\pi}\int_0^\infty \frac{\omega}{a^2+\omega^2}\sin(\omega t)d\omega = e^{-at}.$$

Changing variables ω to x, t to k thus gives the results

$$\int_0^\infty \frac{\cos(kx)}{a^2+x^2}dx = \frac{\pi}{2a}e^{-ak}$$

and

$$\int_0^\infty \frac{x\sin(kx)}{a^2+x^2}dx = \frac{\pi}{2}e^{-ak}.$$

Laplace transforms are extremely useful for finding the solution to differential equations. Fourier transforms can also be so used; however the restrictions on the class of functions allowed is usually prohibitive. Assuming that the improper integrals exist, which requires that $f \to 0$ as $t \to \pm\infty$, let us start with the definition of the Fourier transform

$$F(\omega) = \int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx.$$

Since both limits are infinite, and the above conditions on f hold, we have that the Fourier transform of $f'(t)$, the first derivative of $f(t)$, is straightforwardly $i\omega F(\omega)$ using integration by parts. In general

$$\int_{-\infty}^{\infty} \frac{d^n f}{dt^n} e^{-i\omega t} dt = (i\omega)^n F(\omega).$$

The other principal disadvantage of using Fourier transform is the presence of i in the transform of odd derivatives and the difficulty in dealing with boundary conditions. Fourier sine and cosine transforms are particularly useful however, for dealing with second order derivatives. The results

$$\int_0^{\infty} \frac{d^2 f}{dt^2} \cos(\omega t)dt = -\omega^2 F_c(\omega) - f'(0)$$

and

$$\int_0^{\infty} \frac{d^2 f}{dt^2} \sin(\omega t)dt = -\omega^2 F_s(\omega) + \omega f(0)$$

can be easily derived. These results are difficult to apply to solve differential equations of simple harmonic motion type because the restrictions imposed on $f(t)$ are usually incompatible with the character of the solution. In fact, even when the solution *is* compatible, solving using Fourier transforms is often impractical.

Fourier transforms do however play an important role in solving partial differential equations as will be shown in the next section. Before doing this, we need to square up to some problems involving infinity. One of the common mathematical problems is coping with operations like integrating and differentiating when there are infinities around. When the infinity occurs in the limit of an integral, and there are questions as to whether the integral converges then we need to be particularly careful. This is certainly the case with Fourier transforms of all types; it is less of a problem with Laplace transforms. The following theorem is crucial to this point.

Theorem 6.1 (Lebesgue Dominated Convergence Theorem) *Let*

$$f_h, \quad h \in \mathbb{R}$$

be a family of piecewise continuous functions. If

1. *There exists a function g such that $|f_h(x)| \le g(x)$ $\forall x \in \mathbb{R}$ and all $h \in \mathbb{R}$.*

2.

$$\int_{-\infty}^{\infty} g(x)dx < \infty.$$

3.

$$\lim_{h \to 0} f_h(x) = f(x) \textit{ for every } x \in \mathbb{R}$$

then

$$\lim_{h \to 0} \int_{-\infty}^{\infty} f_h(x)dx = \int_{-\infty}^{\infty} f(x)dx.$$

This theorem essentially tidies up where it is allowable to exchange the processes of taking limits and infinite integration. To see how important and perhaps counter intuitive this theorem is, consider the following simple example. Take a rectangular pulse or "top hat" function defined by

$$f_n(x) = \begin{cases} 1 & n \le x \le n+1 \\ 0 & \text{otherwise.} \end{cases}$$

Let $f(x) = 0$ so that it is true that $\lim_{n \to \infty} f_n(x) = f(x)$ for all $x \in \mathcal{R}$. However by direct integration it is obvious that

$$\lim_{h \to 0} \int_{-\infty}^{\infty} f_n(x)dx = 1 \neq 0 = \int_{-\infty}^{\infty} \lim_{n \to \infty} f_n(x)dx.$$

Thus the theorem does not hold for this function, and the reason is that the function $g(x)$ does not exist. In fact the improper nature of the integrals is incidental.

The proof of the Lebesgue Dominated Convergence Theorem involves classical analysis and is beyond the scope of this book. Suffice it to say that the function $g(x)$ which is integrable over $(-\infty, \infty)$ "dominates" the functions $f_h(x)$, and without it the limit and integral signs cannot be interchanged. As far as we are concerned, the following theorem is closer to home.

Denote by $G\{\mathbb{R}\}$ the family of functions defined on $\mathbb{R}$ with values in $\mathbb{C}$ which are piecewise continuous and absolutely integrable. These are essentially the Fourier transforms as defined in the beginning of Sect. 6.2. For each $f \in G\{\mathbb{R}\}$ the Fourier transform of f is defined for all $\omega \in \mathbb{R}$ by

$$F(\omega) = \int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx$$

as in Sect. 6.2. That $G\{\mathbb{R}\}$ is a linear space over $\mathbb{C}$ is easy to verify. We now state the theorem.

Theorem 6.2 *For each $f \in G\{\mathbb{R}\}$,*

1. *$F(\omega)$ is defined for all $\omega \in \mathbb{R}$.*

2. *F is a continuous function on $\mathbb{R}$.*
3. $\lim\limits_{\omega \pm \infty} F(\omega) = 0.$

Proof To prove this theorem, we first need to use the previous Lebesgue Dominated Convergence Theorem. This was in fact the principal reason for stating it! We know that $|e^{i\omega x}| = 1$, hence

$$\int_{-\infty}^{\infty} |f(x)e^{-i\omega x}|dx = \int_{-\infty}^{\infty} |f(x)|dx < \infty.$$

Thus we have proved the first statement of the theorem; $F(\omega)$ is well defined for every real ω. This follows since the above equation tells us that $f(x)e^{-i\omega x}$ is absolutely integrable on $\mathbb{R}$ for each real ω. In addition, $f(x)e^{-i\omega x}$ is piecewise continuous and so belongs to $G\{\mathbb{R}\}$.

To prove that $F(\omega)$ is continuous is a little more technical. Consider the difference

$$F(\omega + h) - F(\omega) = \int_{-\infty}^{\infty} [f(x)e^{-i\omega(x+h)} - f(x)e^{-i\omega x}]dx$$

so

$$F(\omega + h) - F(\omega) = \int_{-\infty}^{\infty} f(x)e^{-i\omega x}[e^{-i\omega h} - 1]dx.$$

If we let

$$f_h(x) = f(x)e^{-i\omega x}[e^{-i\omega h} - 1]$$

to correspond to the $f_h(x)$ in the Lebesgue Dominated Convergence Theorem, we easily show that

$$\lim_{h\to 0} f_h(x) = 0 \text{ for every } x \in \mathbb{R}.$$

Also, we have that

$$|f_h(x)| = |f(x)||e^{-i\omega x}||e^{-i\omega h} - 1| \le 2|f(x)|.$$

Now, the function $g(x) = 2|f(x)|$ satisfies the conditions of the function $g(x)$ in the Lebesgue Dominated Convergence Theorem, hence

$$\lim_{h\to 0} \int_{-\infty}^{\infty} f_h(x)dx = 0$$

whence

$$\lim_{h\to 0} [F(\omega + h) - F(\omega)] = 0$$

which is just the condition that the function $F(\omega)$ is continuous at every point of $\mathbb{R}$.

The last part of the theorem

$$\lim_{\omega\to\pm\infty} F(\omega) = 0$$

follows from the Riemann–Lebesgue lemma (see Theorem 4.2) since

$$F(\omega) = \int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx = \int_{-\infty}^{\infty} f(x)\cos(\omega x)dx - i\int_{-\infty}^{\infty} f(x)\sin(\omega x)dx,$$
$$\lim_{\omega\to\pm\infty} F(\omega) = 0$$

is equivalent to

$$\lim_{\omega\to\pm\infty}\int_{-\infty}^{\infty} f(x)\cos(\omega x)dx = 0$$

and

$$\lim_{\omega\to\pm\infty}\int_{-\infty}^{\infty} f(x)\sin(\omega x)dx = 0$$

together. These two results are immediate from the Riemann–Lebesgue lemma (see Exercise 5 in Sect. 6.6). This completes the proof. □

The next result, also stated in the form of a theorem, expresses a scaling property. There is no Laplace transform equivalent due to the presence of zero in the lower limit of the integral in this case, but see Exercise 7 in Chap. 1.

Theorem 6.3 *Let $f(x) \in G\{\mathbb{R}\}$ and $a, b \in \mathbb{R}$, $a \neq 0$ and denote the Fourier transform of f by $\mathbb{F}(f)$ so*

$$\mathbb{F}(f) = \int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx.$$

Let $g(x) = f(ax+b)$. Then

$$\mathcal{F}(g) = \frac{1}{|a|}e^{i\omega b/a}\mathcal{F}(f)\left(\frac{\omega}{a}\right).$$

Proof As is usual with this kind of proof, the technique is simply to evaluate the Fourier transform using its definition. Doing this, we obtain

$$\mathcal{F}(g(\omega)) = \int_{-\infty}^{\infty} f(ax+b)e^{-i\omega x}dx.$$

Now simply substitute $t = ax+b$ to obtain, if $a > 0$

$$\mathcal{F}(g(\omega)) = \frac{1}{a}\int_{-\infty}^{\infty} f(t)e^{-i\omega(t-b)/a}dt$$

and if $a < 0$

$$\mathcal{F}(g(\omega)) = -\frac{1}{a}\int_{-\infty}^{\infty} f(t)e^{-i\omega(t-b)/a}dt.$$

So, putting these results together we obtain

$$\mathcal{F}(g(\omega)) = \frac{1}{|a|}e^{i\omega b/a}\mathcal{F}(f)\left(\frac{\omega}{a}\right)$$

as required. □

The proofs of other properties follow along similar lines, but as been mentioned several times already the Fourier transform applies to a restricted set of functions with a correspondingly smaller number of applications.

6.4 Fourier Transforms and Partial Differential Equations

In Chap. 5, two types of partial differential equation, parabolic and hyperbolic, were solved using Laplace transforms. It was noted that Laplace transforms were not suited to the solution of elliptic partial differential equations. Recall the reason for this. Laplace transforms are ideal for solving initial value problems, but elliptic PDEs usually do not involve time and their solution does not yield evolutionary functions. Perhaps the simplest elliptic PDE is Laplace's equation ($\nabla^2\phi = 0$) which, together with ϕ or its normal derivative given at the boundary gives a boundary value problem. The solutions are neither periodic nor are they initial value problems. Fourier transforms as defined so far require that variables tend to zero at $\pm\infty$ and these are often natural assumptions for elliptic equations. There are also two new features that will now be introduced before problems are solved. First of all, partial differential equations such as $\nabla^2\phi = 0$ involve identical derivatives in x, y and, for three dimensional ∇^2, z too. It is logical therefore to treat them all in the same way. This leads to having to define two and three dimensional Fourier transforms. Consider the function $\phi(x, y)$, then let

$$\hat{\phi}(k, y) = \int_{-\infty}^{\infty} \phi(x, y)e^{ikx}dx$$

and

$$\phi_F(k, l) = \int_{-\infty}^{\infty} \hat{\phi}(k, y)e^{ily}dy$$

so that

$$\phi_F(k, l) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \phi(x, y)e^{i(kx+ly)}dxdy$$

becomes a double Fourier transform. In order for these transforms to exist, ϕ must tend to zero uniformly for (x, y) being a large distance from the origin, i.e. as $\sqrt{x^2 + y^2}$ becomes very large. The three dimensional Fourier transform is defined analogously as follows:-

$$\phi_F(k, l, m) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \phi(x, y, z)e^{i(kx+ly+mz)}dxdydz.$$

With all these infinities around, the restrictions on ϕ_F are severe and applications are therefore limited. The frequency ω has been replaced by a three dimensional space (k, l, m) called phase space. However, the kind of problem that gives rise to Laplace's equation does fit this restriction, for example the behaviour of membranes, water or electromagnetic potential when subject to a point disturbance. For this kind of problem, the variable ϕ dies away to zero the further it is from the disturbance, therefore there is a good chance that the above infinite double or triple integral could exist. More useful for practical purposes however are problems in the finite domain and it is these that can be tackled usefully with a modification of the Fourier transform. The unnatural part of the Fourier transform is the imposition of conditions at infinity, and the modifications hinted at above have to do with replacing these by conditions at finite values. We therefore introduce the finite Fourier transform (not to be confused with the FFT—Fast Fourier transform). This is introduced in one dimension for clarity; the finite Fourier transforms for two and three dimensions follow almost at once. If x is restricted to lie between say a and b, then the appropriate Fourier type transformation would be

$$\int_a^b \phi(x)e^{-ikx}dx.$$

This would then be applied to a problem in engineering or applied science where $a \leq x \leq b$. The two-dimensional version could be applied to a rectangle

$$a \leq x \leq b, \quad c \leq y \leq d$$

and is defined by

$$\int_c^d \int_a^b \phi(x, y)e^{-ikx-ily}dxdy.$$

Apart from the positive aspect of eliminating problems of convergence for (x, y) very far from the origin, the finite Fourier transform unfortunately brings a host of negative aspects too. The first difficulty lies with the boundary conditions that have to be satisfied which unfortunately are now no longer infinitely far away. They can and do have considerable influence on the solution. One way to deal with this could be to cover the entire plane with rectangles with $\phi(x, y)$ being doubly periodic, i.e.

$$\phi(x, y) = \phi(x + N(b - a), y + M(d - c)), \quad M, N \text{ integers}$$

then revert to the original infinite range Fourier transform. However, this brings problems of convergence and leads to having to deal with generalised functions. We shall not pursue this kind of finite Fourier transform further here; but for a slightly different approach, see the next section. There seems to be a reverting to Fourier series here, after all the transform was obtained by a limiting process from series, and at first sight, all we seem to have done is reverse the process. A closer scrutiny reveals crucial differences, for example the presence of the factor $1/2\pi$ (or in general $1/l$ where l is the period) in front of the integrals for the Fourier series coefficients a_n and b_n. Much of the care and attention given to the process of letting the limit become infinite involved dealing with the zero that this factor produces. Finite Fourier transforms do not have this factor. We return to some more of these differences in the next section, meanwhile let us do an example.

Example 6.3 *Find the solution to the two-dimensional Laplace equation*

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0, \quad y > 0,$$

with

$$\frac{\partial\phi}{\partial x} \text{ and } \phi \to 0 \text{ as } \sqrt{x^2 + y^2} \to \infty, \quad \phi(x, 0) = 1 \ \ |x| \le 1 \ \ \phi(x, 0) = 0, \quad |x| > 1.$$

Use Fourier transforms in x.

Solution Let

$$\bar{\phi}(k, y) = \int_{-\infty}^{\infty} \phi(x, y) e^{-ikx} dx$$

then

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{\partial^2}{\partial y^2} e^{-ikx} dx &= \frac{\partial^2}{\partial y^2}\left(\int_{-\infty}^{\infty} \phi e^{-ikx} dx\right)\\ &= \frac{\partial^2\bar{\phi}}{\partial y^2}.\end{aligned}$$

Also

$$\int_{-\infty}^{\infty} \frac{\partial^2\phi}{\partial x^2} e^{-ikx} dx$$

can be integrated by parts as follows

$$\int_{-\infty}^{\infty} \frac{\partial^2\phi}{\partial x^2} e^{-ikx} dx = \left[\frac{\partial\phi}{\partial x} e^{-ikx}\right]_{-\infty}^{\infty} + ik\int_{-\infty}^{\infty} \frac{\partial\phi}{\partial x} e^{-ikx} dx$$

$$= ik\left[\phi e^{-ikx}\right]_{-\infty}^{\infty} - k^2\int_{-\infty}^{\infty}\phi e^{-ikx}dx$$
$$= -k^2\bar{\phi}.$$

We have used the conditions that both ϕ and its x derivative decay to zero as $x \to \pm\infty$. Hence if we take the Fourier transform of the Laplace equation in the question,

$$\int_{-\infty}^{\infty}\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right)e^{-ikx}dx = 0$$

or

$$\frac{\partial^2\bar{\phi}}{\partial y^2} - k^2\bar{\phi} = 0.$$

As $\bar{\phi} \to 0$ for large y, the (allowable) solution is

$$\bar{\phi} = Ce^{-|k|y}.$$

Now, we can apply the condition on $y = 0$

$$\begin{aligned}\bar{\phi}(k,0) &= \int_{-\infty}^{\infty}\phi(x,0)e^{-ikx}dx\\ &= \int_{-1}^{1}e^{-ikx}dx\\ &= \frac{e^{ik} - e^{-ik}}{ik}\\ &= \frac{2\sin(k)}{k}\end{aligned}$$

whence

$$C = \frac{2\sin(k)}{k}$$

and

$$\bar{\phi} = \frac{2\sin(k)}{k}e^{-|k|y}.$$

In order to invert this, we need the Fourier transform equivalent of the Convolution theorem (see Chap. 3). To see how this works for Fourier transforms, consider the convolution of the two general functions f and g

$$\begin{aligned}F * G = \int_{-\infty}^{\infty}f(\tau)g(t-\tau)d\tau &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(\tau)G(x)e^{-ix(t-\tau)}d\tau dx\\ &= \int_{-\infty}^{\infty}G(x)e^{-ixt}\int_{-\infty}^{\infty}f(\tau)e^{ix\tau}d\tau dx\end{aligned}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} G(x)F(x)e^{-ixt}dx$$
$$= \frac{1}{2\pi}\mathcal{F}(FG).$$

Now, the Fourier transform of $e^{-|k|y}$ is

$$\frac{y}{k^2+\omega^2}$$

and that of $\phi(x, 0)$ is

$$\frac{2\sin(k)}{k},$$

hence by the convolution theorem,

$$\begin{aligned}\phi(x, y) &= \frac{1}{\pi}y\int_{-1}^{1}\frac{d\tau}{(x-\tau)^2+y^2}\\ &= \frac{1}{\pi}y\left(\tan^{-1}\left(\frac{x-1}{y}\right)+\tan^{-1}\left(\frac{x+1}{y}\right)\right)\end{aligned}$$

is the required solution.

6.5 Windowing

There is no doubt that the most prolific application of Fourier transforms lies in the field of the processing of signals. An in depth analysis is out of place here, but some discussion is helpful as an introduction to the subject of wavelets that follows in the next chapter. To start with, we return to the complex form of the Fourier series and revisit explicitly the close connections between Fourier series and finite Fourier transforms. From Chap. 4 (and Sect. 6.2)

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

with

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx.$$

Thus $2\pi c_n$ is the finite Fourier transform of $f(x)$ over the interval $[-\pi, \pi]$ and the "inverse" is the Fourier series for $f(x)$. If c_n is given as a sequence, then $f(x)$ is easily found. (In practice, the sequence c_n consists of but a few terms.) The resulting $f(x)$ is of course periodic as it is the sum of terms of the type $c_n e^{inx}$ for various n.

Hence the finite Fourier transform of this type of periodic function is a sequence of numbers, usually four or five at most. It is only a short step from this theoretical treatment of finite Fourier transforms to the analysis of periodic signals of the type viewed on cathode ray oscilloscopes. A simple illustrative example follows.

Example 6.4 *Consider the simple "top hat" function $f(x)$ defined by*

$$f(x) = \begin{cases} 1 & x \in [0, \pi] \\ 0 & \textit{otherwise.} \end{cases}$$

Find its finite Fourier transform and finite Fourier sine transform.

Solution The finite Fourier transform of this function is simply

$$\begin{aligned} \int_0^\pi e^{-inx}dx &= \left[-\frac{e^{-inx}}{in}\right]_0^\pi \\ &= \frac{1}{in}(1 - e^{-in\pi}) \end{aligned}$$

which can be written in a number of ways:-

$$\frac{1}{in}(1 - (-1)^n); \quad -\frac{2i}{(2k+1)}; \quad \frac{2\sin(\frac{n\pi}{2})}{n}e^{in\pi/2}.$$

The finite sine transform is a more natural object to find: it is

$$\int_0^\pi \sin(nx)dx = \frac{1}{n}(1 - (-1)^n) = \frac{2}{2k+1}, \quad n, k \text{ integers.}$$

Let us use this example to illustrate the transition from finite Fourier transforms to Fourier transforms proper. The inverse finite Fourier transform of the function $f(x)$ as defined in Example 6.4 is the Fourier series

$$f(x) = \frac{1}{2\pi}\sum_{-\infty}^{\infty}\frac{1 - (-1)^n}{in}e^{inx} \quad 0 \le x \le \pi.$$

However, although $f(x)$ is only defined in the interval $[0, \pi]$, the Fourier series is periodic, period 2π. It therefore represents a square wave shown in Fig. 6.4.

Of course, $x \in [0, \pi]$ so $f(x)$ is represented as a "window" to borrow a phrase from time series and signal analysis. If we write $x = \pi t/l$, then let $l \to \infty$: we regain the transformation that took us from Fourier series to Fourier transforms proper, Sect. 6.2. However, what we have in Fig. 6.5 is a typical signal. The Fourier transform of this signal taken as a whole of course does not exist as conditions at $\pm\infty$ are not satisfied. In the case of an actual signal therefore, the use of the Fourier transform is made possible by restricting attention to a window, that is a finite

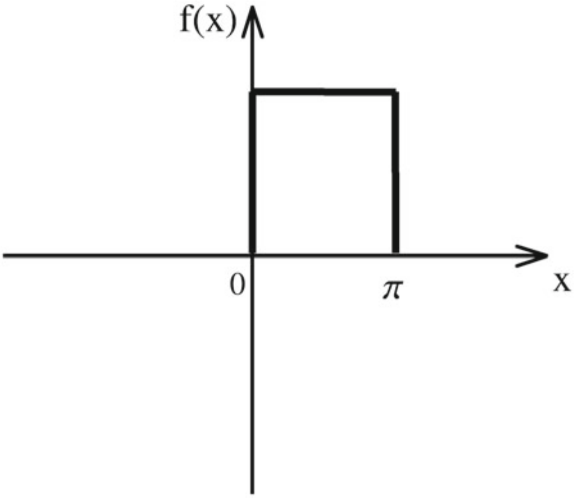


Fig. 6.4 The square wave

range of t. This gives rise to a series (Fourier series) representation of the Fourier transform of the signal. This series has a period which is dictated by the (usually artificially generated) width of the window. The Fourier coefficients give important information on the frequencies present in the original signal. This is the fundamental reason for using these methods to examine signals from such diverse applications as medicine, engineering and seismology. Mathematically, the way forward is through the introduction of the Dirac δ function as follows. We have that

$$\mathcal{F}\{\delta(t-t_0)\} = \int_{-\infty}^{\infty} \delta(t-t_0)e^{-i\omega t}dt = e^{-i\omega t_0}$$

and the inverse result implies that

$$\mathcal{F}\{e^{-it_0 t}\} = 2\pi\delta(\omega-\omega_0).$$

Whence we can find the Fourier transform of a given Fourier series (written in complex exponential form) by term by term evaluation provided such operations are legal in terms of defining the Dirac δ as the limiting case of an infinitely tall but infinitesimally thin rectangular pulse of unit area (see Sect. 2.6).

$$f(t) \sim \sum_{n=-\infty}^{\infty} F_n e^{in\omega_0 t}$$

so that

$$\mathcal{F}\{f(t)\} \sim \mathcal{F}\left\{\sum_{n=-\infty}^{\infty} F_n e^{in\omega_0 t}\right\}$$
$$= \sum_{n=-\infty}^{\infty} F_n \mathcal{F}\{e^{in\omega_0 t}\}$$

which implies

$$\mathcal{F}\{f(t)\} \sim 2\pi \sum_{n=-\infty}^{\infty} F_n \delta(\omega - n\omega_0).$$

Now, suppose we let

$$f(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

that is $f(t)$ is an infinite train of equally spaced Dirac δ functions (called a Shah function by electrical and electronic engineers), then $f(t)$ is certainly periodic (of period T). Of course it is not piecewise continuous, but if we follow the limiting processes through carefully, we can find a Fourier series representation of $f(t)$ as

$$f(t) \sim \sum_{n=-\infty}^{\infty} F_n e^{-in\omega_0 t}$$

where $\omega_0 = 2\pi/T$, with

$$\begin{aligned} F_n &= \frac{1}{T}\int_{-T/2}^{T/2} f(t)e^{-in\omega_0 t}dt \\ &= \frac{1}{T}\int_{-T/2}^{T/2} \delta(t)e^{-in\omega_0 t}dt \\ &= \frac{1}{T} \end{aligned}$$

for all n. Hence we have the result that

$$\begin{aligned} \mathcal{F}\{f(t)\} &= 2\pi \sum_{n=-\infty}^{\infty} \frac{1}{T}\delta(\omega - n\omega_0) \\ &= \omega_0 \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_0). \end{aligned}$$

Which means that the Fourier transform of an infinite string of equally spaced Dirac δ functions (Shah function) is another string of equally spaced Dirac δ functions:

$$\mathcal{F}\left\{\sum_{n=-\infty}^{\infty}\delta(t-nT)\right\}=\omega_0\sum_{n=-\infty}^{\infty}\delta(\omega-n\omega_0).$$

It is this result that is used extensively by engineers and statisticians when analysing signals using sampling. Mathematically, it is of interest to note that with $T = 2\pi$ $(\omega_0 = 1)$ we have found an invariant under Fourier transformation.

If $f(t)$ and $F(\omega)$ are a Fourier transform pair, then the quantity

$$E=\int_{-\infty}^{\infty}|f(t)|^2dt$$

is called the *total energy*. This expression is an obvious carry over from $f(t)$ representing a time series. (Attempts at dimensional analysis are fruitless due to the presence of a dimensional one on the right hand side. This is annoying to physicists and engineers, very annoying!) The quantity $|F(\omega)|^2$ is called the *energy spectral density* and the graph of this against ω is called the *energy spectrum* and remains a very useful guide as to how the signal $f(t)$ can be thought of in terms of its decomposition into frequencies. The energy spectrum of $\sin(kt)$ for example is a single spike in ω space corresponding to the frequency $2\pi/k$. The constant energy spectrum where all frequencies are present in equal measure corresponds to the "white noise" signal characterised by a hiss when rendered audible. The two quantities $|f(t)|^2$ and energy spectral density are connected by the transform version of Parseval's theorem (sometimes called Rayleigh's theorem, or Plancherel's identity). See Theorem 4.8 for the series version.

Theorem 6.4 (Parseval's, for transforms) *If $f(t)$ has a Fourier transform $F(\omega)$ and*

$$\int_{-\infty}^{\infty}|f(t)|^2dt<\infty$$

then

$$\int_{-\infty}^{\infty}|f(t)|^2dt=\frac{1}{2\pi}\int_{-\infty}^{\infty}|F(\omega)|^2d\omega.$$

Proof The proof is straightforward:-

$$\begin{aligned}\int_{-\infty}^{\infty}f(t)f^*(t)dt&=\int_{-\infty}^{\infty}f(t)\frac{1}{2\pi}\int_{-\infty}^{\infty}F(\omega)e^{i\omega t}d\omega dt\\&=\frac{1}{2\pi}\int_{-\infty}^{\infty}F(\omega)\int_{-\infty}^{\infty}f(t)e^{i\omega t}dtd\omega\\&=\frac{1}{2\pi}\int_{-\infty}^{\infty}F(\omega)(F(\omega))^*d\omega\\&=\frac{1}{2\pi}\int_{-\infty}^{\infty}|F(\omega)|^2d\omega\end{aligned}$$

where f^* is the complex conjugate of $f(t)$. The exchange of integral signs is justified as long as their values remain finite. □

Most of the applications of this theorem lie squarely in the field of signal processing, but here is a simple example using the definition of energy spectral density.

Example 6.5 *Determine the energy spectral densities for the following functions:*

(i)

$$f(t) = \begin{cases} A & |t| < T \\ 0 & otherwise \end{cases}$$

This is the same function as in Example 6.2.

(ii)

$$f(t) = \begin{cases} e^{-at} & t \geq 0 \\ 0 & t < 0. \end{cases}$$

Solution The energy spectral density $|F(\omega)|^2$ is found from $f(t)$ by first finding its Fourier transform. Both calculations are essentially routine.

(i)

$$\begin{aligned} F(\omega) &= \int_{-\infty}^{\infty} f(t)e^{i\omega t}dt \\ &= A\int_{-T}^{T} e^{i\omega t}dt \\ &= \frac{A}{i\omega}\left[e^{i\omega t}\right]_{-T}^{T} \\ &= \frac{A}{i\omega}[e^{i\omega T} - e^{-i\omega T}] = \frac{2A\sin(\omega t)}{\omega} \end{aligned}$$

as already found in Example 6.2. So we have that

$$|F(\omega)|^2 = \frac{4A^2\sin^2(\omega T)}{\omega^2}.$$

(ii)

$$\begin{aligned} F(\omega) &= \int_{0}^{\infty} e^{-at}e^{-i\omega t}dt \\ &= \left[-\frac{e^{(-a-i\omega)t}}{a+i\omega}\right]_0^{\infty} \\ &= \frac{a-i\omega}{a^2+\omega^2}. \end{aligned}$$

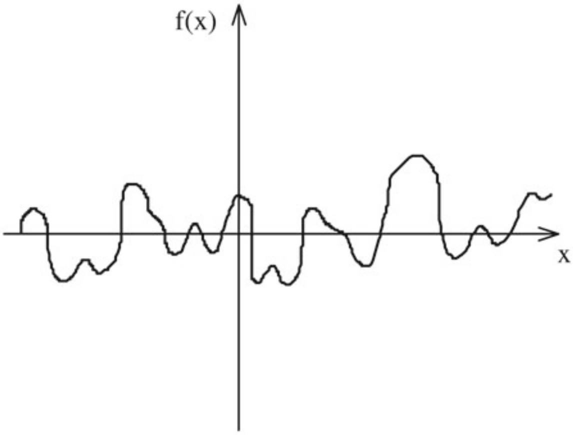


Fig. 6.5 A time series

Hence

$$|F(\omega)|^2 = \frac{a - i\omega}{a^2 + \omega^2} \cdot \frac{a + i\omega}{a^2 + \omega^2} = \frac{1}{a^2 + \omega^2}.$$

There is another aspect of signal processing that ought to be mentioned. Most signals are not deterministic and have to be analysed by using statistical techniques such as sampling. It is by sampling that a time series which is given in the form of an analogue signal (a wiggly line as in Fig. 6.5) is transformed into a digital one (usually a series of zeros and ones).

The Fourier transform is a means by which a signal can be broken down into component frequencies, from t space to ω space. This cannot be done directly since time series do not conveniently obey the conditions at $\pm\infty$ that enable the Fourier transform to exist formally. The autocovariance function is the convolution of f with itself and is a measure of the agreement (or correlation) between two parts of the signal time t apart. It turns out that the autocovariance function of the time series is, however well behaved at infinity and it is usually this function that is subject to spectral decomposition either directly (analogue) or via sampling (digital). Digital time series analysis is now a very important subject due to the omnipresent (digital) computer. Indeed all television is now digital as are mobile communications. We shall not pursue practical implimentation of digital signal analysis further here. What we hope to have achieved is an appreciation of the importance of Fourier transforms and Fourier series to the subject.

Let us finish this chapter by doing an example which demonstrates a slightly different way of illustrating the relationship between finite and standard Fourier transforms.

The electrical engineering fraternity define the window function by $W(x)$ where

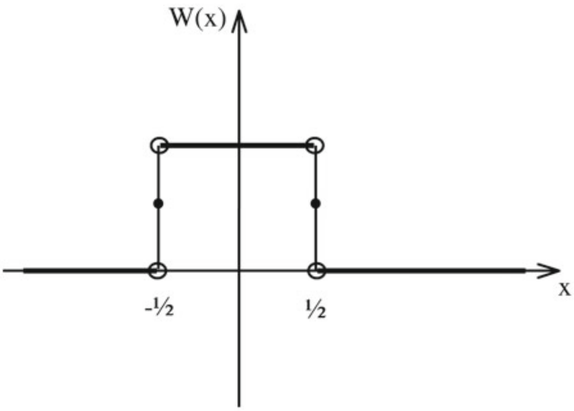


Fig. 6.6 The window function $W(x)$

$$W(x) = \begin{cases} 0 & |x| > \frac{1}{2} \\ \frac{1}{2} & |x| = \frac{1}{2} \\ 1 & |x| < \frac{1}{2}. \end{cases}$$

giving the picture of Fig. 6.6. This is almost the same as the "top hat" function defined in the last example. Spot the subtle difference.

Use of this function in Fourier transforms immediately converts a Fourier transform into a finite Fourier transform as follows

$$\int_{-\infty}^{\infty} W\left(\frac{x - \frac{1}{2}(b+a)}{b-a}\right) f(x)e^{-i\omega x}dx = \int_a^b f(x)e^{-i\omega x}dx.$$

It is easy to check that if

$$t = \frac{x - \frac{1}{2}(b+a)}{b-a}$$

then $t > \frac{1}{2}$ corresponds to $x > b$ and $t < -\frac{1}{2}$ corresponds to $x < a$. What this approach does is to move the work from inverting a finite Fourier transform in terms of Fourier series to evaluating

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} F_W(\omega)e^{i\omega x}d\omega$$

where $F_W(\omega)$ is the Fourier transform of the "windowed" version of $f(x)$. It will come as no surprise to learn that the calculation of this integral is every bit as difficult (or easy) as directly inverting the finite Fourier transform. The choice lies between working with Fourier series directly or working with $F_W(\omega)$ which involves series of generalised functions.

6.6 Exercises

1. Determine the Fourier transform of the function $f(t)$ defined by

$$f(t) = \begin{cases} T+t & -T \le t < 0 \\ T-t & 0 \le t < T \\ 0 & \text{otherwise.} \end{cases}$$

2. If $f(t) = e^{-t^2}$, find its Fourier transform.
3. Show that

$$\int_0^\infty \frac{\sin(u)}{u} du = \frac{\pi}{2}.$$

4. Define

$$f(t) = \begin{cases} e^{-t} & t \ge 0 \\ 0 & t < 0 \end{cases}$$

and show that

$$f(at) * f(bt) = \frac{f(at) - f(bt)}{b-a}$$

where a and b are arbitrary real numbers. Hence also show that $f(at) * f(at) = tf(at)$ where $*$ is the Fourier transform version of the convolution operation defined by

$$f(t) * g(t) = \int_{-\infty}^\infty f(\tau)g(t-\tau)d\tau.$$

5. Consider the integral

$$g(x) = \int_{-\infty}^\infty f(t)e^{-2\pi ixt} dt$$

and, using the substitution $u = t - 1/2x$, show that $|g(x)| \to 0$ hence providing a simple illustration of the Riemann–Lebesgue lemma.
6. Derive Parseval's formula:-

$$\int_{-\infty}^\infty f(t)G(it)dt = \int_{-\infty}^\infty F(it)g(t)dt$$

where F and G are the Fourier transforms of $f(t)$ and $g(t)$ respectively and all functions are assumed to be well enough behaved.

7. Define $f(t) = 1 - t^2, \quad -1 < t < 1$, zero otherwise, and $g(t) = e^{-t}, \quad 0 \le t < \infty$, zero otherwise. Find the Fourier transforms of each of these functions and hence deduce the value of the integral

$$\int_0^\infty \frac{4e^{-t}}{t^3}(t\cosh(t) - \sinh(t))dt$$

by using Parseval's formula (see Exercise 6). Further, use Parseval's theorem for Fourier transforms, Theorem 6.9, to evaluate the integral

$$\int_0^\infty \frac{(t\cos(t) - \sin(t))^2}{t^6}dt$$

8. Consider the partial differential equation

$$u_t = ku_{xx} \quad x > 0, \quad t > 0$$

with boundary conditions $u(x, 0) = g(x), \quad u(0, t) = 0$, where $g(x)$ is a suitably well behaved function of x. Take Laplace transforms in t to obtain an ordinary differential equation, then take Fourier transforms in x to solve this in the form of an improper integral.
9. The Helmholtz equation takes the form $u_{xx} + u_{yy} + k^2u = f(x, y)$ $-\infty < x, y < \infty$. Assuming that the functions $u(x, y)$ and $f(x, y)$ have Fourier transforms show that the solution to this equation can formally be written:-

$$u(x, y) = -\frac{1}{4\pi^2}\int\int\int\int e^{-i[\lambda(x-\xi)+\mu(y-\eta)]}\frac{f(\xi, \eta)}{\lambda^2 + \mu^2 - k^2}d\lambda d\mu d\xi d\eta,$$

where all the integrals are from $-\infty$ to ∞. State carefully the conditions that must be obeyed by the functions $u(x, y)$ and $f(\xi, \eta)$.
10. Use the window function $W(x)$ defined in the last section to express the coefficients (c_n) in the complex form of the Fourier series for an arbitrary function $f(x)$ in terms of an integral between $-\infty$ and ∞. Hence, using the inversion formula for the Fourier transform find an integral for $f(x)$. Compare this with the Fourier series and the derivation of the transform in Sect. 6.2, noting the role of periodicity and the window function.
11. The two series

$$F_k = \sum_{n=0}^{N-1} f_n e^{-ink\Delta\omega T},$$

$$f_n = \frac{1}{N}\sum_{k=0}^{N-1} F_k e^{ink\Delta\omega T}$$

where $\Delta\omega = 2\pi/NT$ define the *discrete* Fourier transform and its inverse. Outline how this is derived from continuous (standard) Fourier transforms by considering the series f_n as a sampled version of the time series $f(t)$.

12. Using the definition in the previous exercise, determine the discrete Fourier transform of the sequence $\{1, 2, 1\}$ with $T = 1$.
13. Find the Fourier transform of the Dirac δ function $\delta(t-t_0)$ for arbitrary t_0. Hence express $\delta(t)$ as the integral of an exponential.
14. Find, in terms of Dirac δ functions the Fourier transforms of $\cos(\omega_0 t)$ and $\sin(\omega_0 t)$, where ω_0 is a constant.

Chapter 7
Wavelets and Signal Processing

7.1 Introduction

In this chapter, more modern notions relevant to signal processing are introduced. The Fourier series of Chap. 4 and Fourier transform of Chap. 6 are taken to the next step, the idea being to use the power of transforming from time to frequency in order to analyse what are in effect complicated functions of time called time series. We shall only give an introduction to signal processing as there are very large tomes devoted to the subject.

7.2 Wavelets

In Chap. 6 we saw that one problem with Fourier transforms was convergence. Straight from the definition of Fourier transform, many everyday functions simply do not possess Fourier transforms as the integrals diverge. By extending the meaning of integration to incorporate generalised functions some crucial elementary functions were shown to possess Fourier transforms, but in the practical analysis of time series this is not enough and something else needs to be done. There are two possibilities. One trick used is to truncate the time series, that is, start the time series at say $-T/2$ and finish it at $T/2$ for some fixed usually large time T. Another is more subtle and was introduced very briefly in the last chapter; define a different quantity called the autocovariance (or its normalised version, autocorrelation) which is a measure of agreement between the signal at a particular time and its value t seconds later, defined mathematically as an integral very much like the convolution of the function with itself over a large range of time divided by this time. This dies away as $t \to -\infty$ and $t \to \infty$ straight from its definition, and it is easy to prove rigorously. The Fourier transform of this is called the spectral density (or power spectral density) and it is this that is used to analyse signals. Considering both alternatives, it is the first of these that is relevant to the study of wavelets, and is what we concentrate on first

P. Dyke, *An Introduction to Laplace Transforms and Fourier Series*, Springer Undergraduate Mathematics Series, DOI: 10.1007/978-1-4471-6395-4_7,

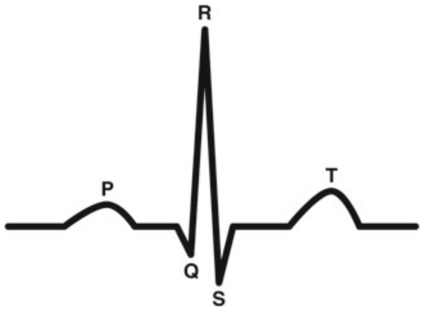


Fig. 7.1 A single heartbeat shown using an electrocardiograph (or ECG)

in this chapter. More will then be said on the second of these approaches. However, not much of practical value can be done without the introduction of statistical and probabilistic concepts. These are out of place in a text on Laplace transforms and Fourier series and the interested reader is directed towards specialist books on signal processing for technicalities beyond those done here. The next few sections will principally be concerned with functions that are zero outside a range of values of t. The easiest way to think of a wavelet is as a pulse of the type seen when the heart is monitored on an ECG (electrocardiogram) see Fig. 7.1. This figure shows the four different pulses labelled P, Q, R, S and T that comprise the typical heartbeat. The point being that the whole beat though complex is constrained to lie within a short time interval. Mathematically, perhaps the sinc function is the idealised wavelet:

$$\operatorname{sinc}(t) = \frac{\sin(t)}{t}$$

(truncated to lie between $\pm 2\pi$ say, shown in Fig. 7.2) but there are plenty of others. If t is replaced by πt the function is sometimes called the Shannon function. The idea now is to proceed formally and from this idea of a single pulse of a specific shape, find a linear space so that in the same way as in Fourier series, a general signal can be generated from a basis of pulse functions. Appendix C will need to be consulted by those less acquainted with such mathematical formalities.

7.3 Basis Functions

It is reasonable here to repeat some basic notions relevant to linear spaces first encountered in Chap. 4. The idea is that for a certain set of functions $\phi_k(t)$ it is possible to express any arbitrary function $f(t)$ as a linear combination

$$f(t) = \sum_{k=-\infty}^{\infty} c_k \phi_k(t)$$

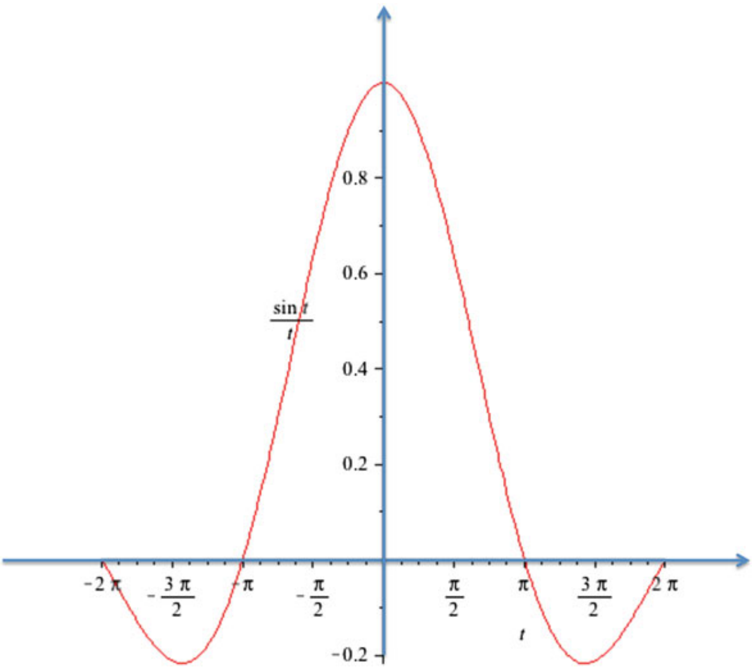


Fig. 7.2 The sinc function truncated at $\pm 2\pi$

where straight away we have assumed that the number of elements in the basis is infinite, in fact a double infinity. The linear space is assumed to possess an inner product of the type:

$$\langle f, \phi_k \rangle = \int_{-\infty}^{\infty} f(t)\overline{\phi_k(t)}dt$$

moreover the basis functions themselves will be assumed to obey an orthogonality relationship:

$$\langle \phi_k, \phi_l \rangle = \int_{-\infty}^{\infty} \phi_k(t)\overline{\phi_l(t)}dt = \delta_{k,l}.$$

Here c_k is a constant coefficient, $\delta_{k,l}$ is the Kronecker delta defined by:

$$\delta_{k,l} = \begin{cases} 1, & k = l \\ 0, & k \neq l \end{cases}$$

and finally all the functions $\phi_k(t)$ are presumed to belong to the set of all square integrable functions that is:

$$\int_{-\infty}^{\infty} \{f(t)\}^2 dt < \infty.$$

The overbar denotes the complex conjugate. For the case of real functions it can of course be ignored. In the above the basis functions are in fact orthonormal, but this is not an essential requirement.

Perhaps the simplest way to generate a set of basis functions is to start with the Haar function, named after a Hungarian mathematician Alfréd Haar (1885–1933) who introduced it in 1910. This can be thought of as our old friend the Heaviside step function (see Sect. 2.1) but restricted in the way of wavelets to lie between 0 and 1. However, some books insist that even the most basic Haar function should integrate to zero, that is the area above the t axis is equal to the area below. However, here let us stick to the simplest definition of the Haar characteristic function as follows:

$$\chi_{[0,1)}(t) = \begin{cases} 1, & 0 \le t < 1 \\ 0, & \text{otherwise} \end{cases}$$

where the notation is as follows: $\chi_{(j,k)}$ means that its value is 1 between j and k but 0 elsewhere. The usual set notation applies whereby a square bracket includes the endpoint(s) whereas a parenthesis doesn't. It is sometimes called a top hat function as this is what the graph resembles. Now we define

$$\phi_k(t) = \chi_{[0,1)}(t-k), \qquad k \in \mathbb{Z}$$

so that the orthogonality relationship:

$$\langle \phi_k, \phi_l \rangle = \int_{-\infty}^{\infty} \chi_{[0,1)}(t-j)\overline{\chi_{[0,1)}(t-k)}dt = \delta_{j,k}$$

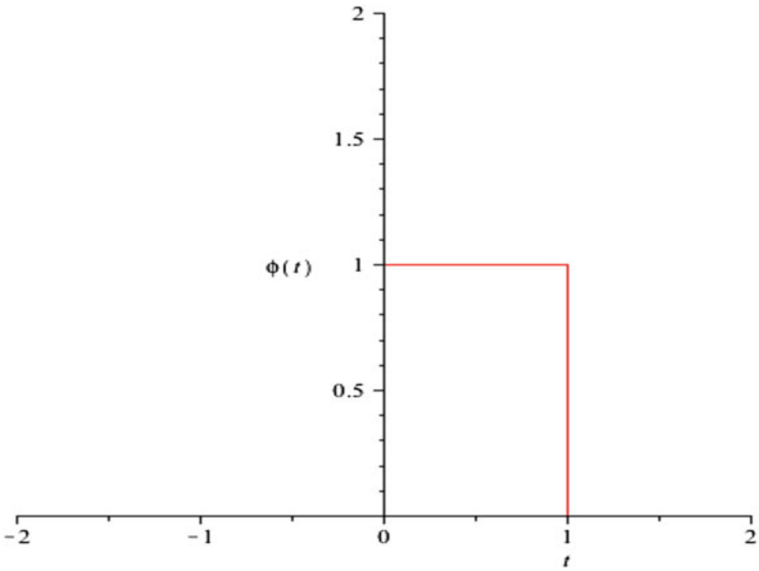
holds. When $j \neq k$ ($j, k \in \mathbb{Z}$) there is no overlap between the top hats so the integral is zero. If $j = k$ the top hats precisely coincide and the area under the curve is unity. Having established the orthogonality of this basis of Haar functions, we are able to generate many others. From the main function we can define a whole family of functions. The starting function is the *father* an associated function will be the *mother*, and there are functions derived from them called *sons* and *daughters*, textbooks devoted to wavelets talk of *sibling rivalry* and so the family analogy continues. If the father is the Haar function the associated Haar mother wavelet function is defined by:

$$\psi(t) = \begin{cases} 1, & 0 \le t < \frac{1}{2} \\ -1, & \frac{1}{2} \le t < 1 \\ 0, & \text{otherwise.} \end{cases}$$

Both $\phi(t)$ and $\psi(t)$, are shown in Fig. 7.3. The relation between the father wavelet $\phi(t)$ and mother wavelet $\psi(t)$ is

$$\psi(t) = \phi(2t) - \phi(2t-1)$$

so the first generation of daughters are defined



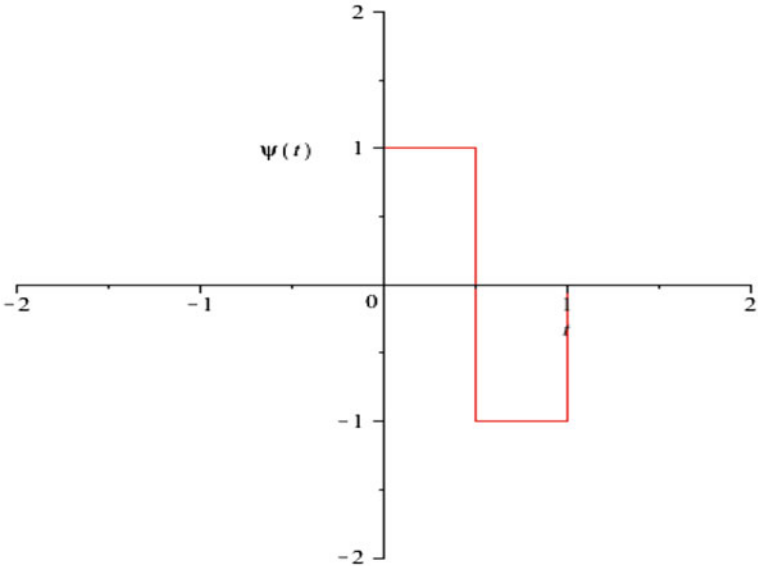


Fig. 7.3 The $\phi(t)$ (*above*) and $\psi(t)$ functions from which the father and mother wavelets arise

$$\psi_{1,0}(t) = \psi(2t) = \phi(4t) - \phi(4t - 1), \quad \psi_{1,1}(t) = \phi(2t - 1) = \phi(4t - 2) - \phi(4t - 3).$$

The second of these daughter wavelets is shown below: This terminology, swathed as it is in family analogies seems quaint but it is now accepted as standard. From the definitions, or directly from the figures, we have that:

$$\psi(2t) = \psi_{1,0} = \begin{cases} 1, & 0 \le t < \frac{1}{4} \\ -1, & \frac{1}{4} \le t < \frac{1}{2} \\ 0, & \text{otherwise} \end{cases}$$

and also

$$\psi(2t-1) = \psi_{1,1} = \begin{cases} 1, & \frac{1}{2} \le t < \frac{3}{4} \\ -1, & \frac{3}{4} \le t < 1 \\ 0, & \text{otherwise.} \end{cases}$$

Defined in this way, only the basic Haar function does not integrate to zero, all of the rest of the family do.

Example 7.1 *Using the Father, Mother, Daughter1 and −Daughter1 wavelets defined above, is it possible to decompose the signals*

$$g_1(t) = \begin{cases} 4, & 0 \le t < \frac{1}{4} \\ -6, & \frac{1}{4} \le t < \frac{1}{2} \\ 3, & \frac{1}{2} \le t \le \frac{3}{4} \\ 2, & \frac{3}{4} \le t \le 1 \\ 0, & \textit{otherwise} \end{cases}$$

and

$$g_2(t) = \begin{cases} 4, & 0 \le t < \frac{1}{4} \\ 4, & \frac{1}{4} \le t < \frac{1}{2} \\ 3, & \frac{1}{2} \le t \le \frac{3}{4} \\ 2, & \frac{3}{4} \le t \le 1 \\ 0, & \textit{otherwise} \end{cases}$$

by expressing it as a linear combination of ϕ, ψ, $\psi_{1,1}$, $-\psi_{1,1}$? Give reasons for your answers.

Solution If we simply proceed to try and solve the vector equation:

$$a\phi + b\psi + c\psi_{1,1} - d\psi_{1,1} = (4, -6, 3, 2)^T$$

we get the four simultaneous equations:

$$\begin{aligned} a + b &= 4 \\ a + b &= -6 \\ a - b + c - d &= 3 \\ a - b - c + d &= 2 \end{aligned}$$

and we immediately see that the first two are inconsistent as 4 and −6 cannot be equal. So the answer to the first part is, no, $g_1(t)$ cannot be expressed in terms of the four wavelets $\phi, \psi, \psi_{1,0}, \psi_{1,1}$. The second function gives the equation

$$a\phi + b\psi + c\psi_{1,1} - d\psi_{1,1} = (4, 4, 3, 2)^T$$

and the four simultaneous equations

$$\begin{aligned} a + b &= 4 \\ a + b &= 4 \\ a - b + c - d &= 3 \\ a - b - c + d &= 2. \end{aligned}$$

This time the first two equations are identical and there is no inconsistency, so there is a solution to the problem. However there are an infinity of them:

$$a = 3.25, b = 0.75, c = \lambda, d = 0.5 + \lambda$$

where λ can take any arbitrary value. Therefore yes $g_2(t)$ can be expressed in terms of the four wavelets $\phi, \psi, \psi_{1,1}, -\psi_{1,1}$ but not uniquely. Of course the problem here is that the set $\phi, \psi, \psi_{1,1}, -\psi_{1,1}$ does not constitute a basis for functions of the type $g_1(t)$ or $g_2(t)$ and this example shows why we need to have a basis. We have also assumed a mapping (bijective mapping) between $g_1(t)$ and the vector $(4, -6, 3, 2)$ similarly for $g_2(t)$ and $(4, 4, 3, 2)$. This leads us nicely to the next section when all these points are addressed (Fig. 7.4).

7.4 The Four Wavelet Case

One of the striking characteristics of wavelets that is also their prime usefulness is their simple and limited range. Wavelets are zero outside a window, but can have any value inside the window. For example to simulate the heartbeat pulse of Fig. 7.1

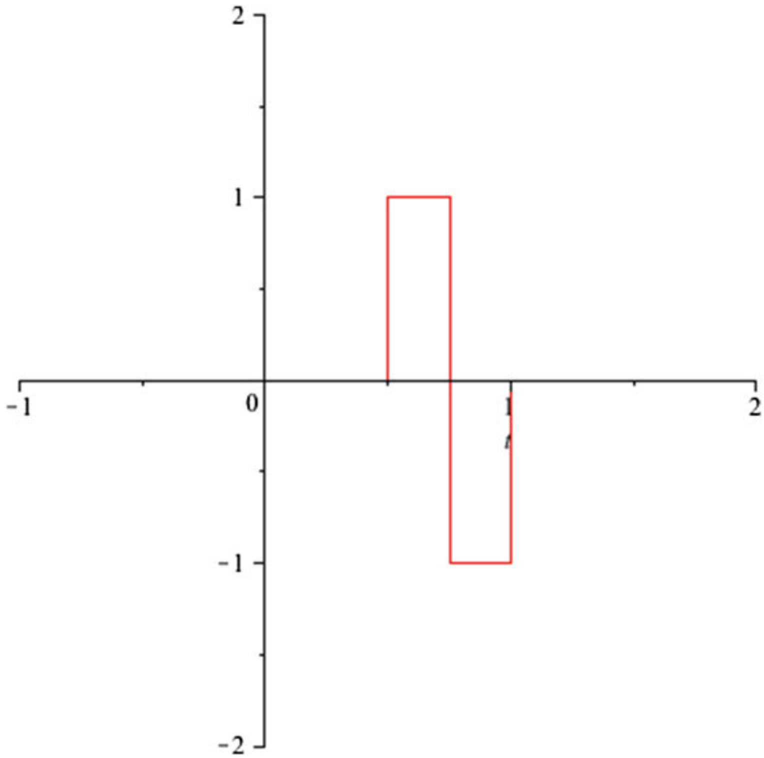


Fig. 7.4 The daughter wavelet $\psi_{1,1}(t)$

one would imagine wavelets with linear dependence on time (saw tooth) would be required. Suppose for now the values are piecewise constant, and within that window wavelets have at most four values. This means that identifying each wavelet with a four vector is possible, and so we can address the difficulties encountered in Example 7.1. As hinted at there, this is done through a mapping. A mapping that is an isomorphism (or bijection). This collection of four vectors then can be subject to the mathematics of vector spaces which brings out useful features and enables calculation. First of all, split the interval [0, 1] into four equal parts: [0, 1/4], [1/4, 1/2], [1/2, 3/4] and [3/4, 1] then identify the wavelet functions as follows:

$$\phi \longleftrightarrow \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad \psi \longleftrightarrow \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} \quad \psi_{1,0} \longleftrightarrow \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} \psi_{1,1} \longleftrightarrow \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}.$$

In general, the function $f(t)$ defined below corresponds to the four vector on the right through a bijective mapping.

$$f(t) = \begin{cases} a, \; 0 \le t < \dfrac{1}{4} \\ b, \; \dfrac{1}{4} \le t < \dfrac{1}{2} \\ c, \; \dfrac{1}{2} \le t < \dfrac{3}{4} \\ d, \; \dfrac{3}{4} \le t < 1 \\ 0, \; \text{otherwise} \end{cases} \longleftrightarrow \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}.$$

Having identified this bijective mapping, the well known properties of the vector space come to our aid in much the same way as they did in Fourier series. For example the four unit vectors:

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

are a basis for all four vectors

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}.$$

Thus any wavelet function $f(t)$ can be generated by the four simple top hat functions $\gamma_i(t), i = 1, 2, 3, 4$ where

$$\phi_{2,0}(t) = \begin{cases} 1, & 0 \le t < \dfrac{1}{4} \\ 0, & \text{otherwise,} \end{cases}$$

$$\phi_{2,1}(t) = \begin{cases} 1, & \dfrac{1}{4} \le t < \dfrac{1}{2} \\ 0, & \text{otherwise,} \end{cases}$$

$$\phi_{2,2}(t) = \begin{cases} 1, & \dfrac{1}{2} \le t < \dfrac{3}{4} \\ 0, & \text{otherwise,} \end{cases}$$

and

$$\phi_{2,3}(t) = \begin{cases} 1, & \dfrac{3}{4} \le t < 1 \\ \\ 0, & \text{otherwise.} \end{cases}$$

in much the same way as any periodic function can be generated by a combination of sines and cosines. The functions $\phi_{2,0}, \phi_{2,1}, \phi_{2,2}, \phi_{2,3}$ are called the son wavelet functions; this will be generalised later. As convenient as this may seem, it turns out that the basis $\phi_{2,0}, \phi_{2,1}, \phi_{2,2}, \phi_{2,3}$ is not the only one used for the purposes of calculation. Instead it is usually best to use the ones introduced earlier through the Haar mother wavelet function. We still start with the father, but then choose the mother, daughter1 and daughter2 functions. So we have the basis $\phi(t), \psi(t), \psi_{1,0}(t), \psi_{1,1}(t)$ which map to the four vectors

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \text{ and } \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}$$

respectively.

The most straightforward way of generalising this, is to focus on the division of the interval [0, 1] by twos. The above example divided this interval into quarters; so let us define 2^j piecewise constant wavelets and identify column j vectors as above. This can be done by defining $\psi_j(t)$ as the mother wavelet through

$$\psi_j(t) = 2^{j/2}\psi\left(2^j t\right) \tag{7.1}$$

where j is an integer. The generalisation to the kth generation of daughter wavelets are found through the formula:

$$\psi_{j,k}(t) = 2^{j/2}\psi\left(2^j t - k\right) \tag{7.2}$$

where $0 \le k \le 2^j - 1$. This function has the value 1 in the interval $[k2^{-j}, (k + \frac{1}{2})2^{-j}]$ and the value -1 in the adjacent interval $[(k + \frac{1}{2})2^{-j}, (k + 1)2^{-j}]$. It is zero everywhere else, so has integral 0. They are also pair-wise orthogonal:

$$\int_{-\infty}^{\infty} \psi_{j,k}(t)\psi_{p,q}(t)dt = \delta_{j,p}\delta_{k,q},$$

the right hand side being the product of two Kronecker delta functions which is zero unless $j = p$ and $k = q$ in which case it is one.

Then defining mother, daughters, grand-daughters, great grand-daughters etc. depending on the values of j and k. There will be 2^j members of the basis corresponding to j generations. There is a strong correlation between computation using wavelets defined like this and the Fast Fourier Transform or FFT (see the book by Stéphane Mallet "A Wavelet Tour of signal processing: The Sparse Way").

If $f(t)$ represents a signal that can be interpreted in terms of a series of such wavelets:

$$f(t) = \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} c_{jk}\psi_{j,k}$$

then define an inner product through

$$\langle f, \psi_j \rangle = \int_{-\infty}^{\infty} f(t)\psi_{j,k}(t)dt$$

so that using the orthogonal properties of the basis, the function $f(t)$ can be recovered through

$$f(t) = \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \langle f, \psi_{j,k} \rangle \psi_{j,k}$$

which, if you unpick the notation, is a generalisation of a Fourier series and the integrals that provide formulas for the coefficients. Here, the overbar has been omitted as all variables and functions are assumed real valued. This is by no means the only way to generalise the Haar wavelets and its generations, but it is one of the simplest.

Example 7.2 *Define explicitly the next generation of wavelet functions, corresponding to Father, Mother, Daughter1, Daughter2, Granddaughter1, Granddaughter2, Granddaughter3 and Granddaughter4.*

Solution The Father, Mother and Daughter wavelets will be the same, except of course this generation will have functions isomorphic to vectors with eight ($2^{j+1} = 2^3$ so $j = 2$) elements. By construction, the vectors that represent the eight wavelets are:

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ -1 \\ -1 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}.$$

So the wavelet functions are $\phi(t)$, $\psi(t)$, $\psi_{0,1}(t)$ and $\psi_{1,1}$ together with the following four granddaughter wavelets:

$$\psi_{2,0} = \begin{cases} 1, & 0 \le t < \dfrac{1}{8} \\ -1, & \dfrac{1}{8} \le t < \dfrac{1}{4} \\ 0, & \text{otherwise} \end{cases}$$

$$\psi_{2,1} = \begin{cases} 1, & \dfrac{1}{4} \le t < \dfrac{3}{8} \\ -1, & \dfrac{3}{8} \le t < \dfrac{1}{2} \\ 0, & \text{otherwise} \end{cases}$$

$$\psi_{2,2} = \begin{cases} 1, & \dfrac{1}{2} \le t < \dfrac{5}{8} \\ -1, & \dfrac{5}{8} \le t < \dfrac{3}{4} \\ 0, & \text{otherwise} \end{cases}$$

$$\psi_{2,3} = \begin{cases} 1, & \dfrac{3}{4} \le t < \dfrac{7}{8} \\ -1, & \dfrac{7}{8} \le t < 1 \\ 0, & \text{otherwise.} \end{cases}$$

One can see at a glance that the eight column vectors have a zero scalar product and so are orthogonal. This then tells us that the functions that they represent are also orthogonal. This can be deduced by direct integration, though it takes a little longer. The vectors are not orthonormal of course, but they can be rendered so upon multiplication by the reciprocal of square root of the sum of the squares of each component. So division by $\sqrt{8}$, $\sqrt{8}$, 2, 2, $\sqrt{2}$, $\sqrt{2}$, $\sqrt{2}$, and $\sqrt{2}$ respectively will give a set of eight orthonormal vectors that represent the functions which form a basis for all functions that are piecewise constant over the eight intervals $[0, \frac{1}{8}]$, $[\frac{1}{8}, \frac{1}{4}]$, $[\frac{1}{4}, \frac{3}{8}]$, $[\frac{3}{8}, \frac{1}{2}]$, $[\frac{1}{2}, \frac{5}{8}]$, $[\frac{5}{8}, \frac{3}{4}]$, $[\frac{3}{4}, \frac{7}{8}]$ and $[\frac{7}{8}, 1]$. All of these are of course special cases of Eq. (7.2). You should verify this as an exercise.

With all this concentration on daughter wavelets, one can be forgiven asking whether we should use son wavelets. The generalisation of the son wavelet is achieved through the original Haar wavelet $\phi(t)$ through

$$\phi_{j,k}(t) = \phi(2^j t - k)$$

though these are not orthonormal so multiplication by the factor $2^{j/2}$ can be done. Whether one chooses daughter or son wavelets is called sibling rivalry, however the daughter wavelets are usually preferred as they have zero mean.

7.5 Transforming Wavelets

There are many other kinds of wavelets, but the moment has arrived when we need to find out what happens when we transform them. As a first example let us find the Fourier transform of $\psi(t)$. Recalling the definition of Fourier transform (Chap. 6):

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt$$

where the trivial change of variable $x = t$ has been made as it is always the case that transforms are performed in the time domain where wavelets are concerned. We present this formally as an example:

Example 7.3 *Find the Fourier transform of the mother wavelet function $\psi(t)$.*

Solution The definition of the mother wavelet $\psi(t)$ was given earlier, and is:

$$\psi(t) = \begin{cases} 1, & 0 \le t < \frac{1}{2} \\ -1, & \frac{1}{2} \le t < 1 \\ 0, & \text{otherwise.} \end{cases}$$

Thus

$$\begin{aligned} F_\psi(\omega) &= \int_{-\infty}^{\infty} \psi(t)e^{-i\omega t}dt \\ &= \int_0^{\frac{1}{2}} e^{-i\omega t}dt - \int_{\frac{1}{2}}^{0} e^{-i\omega t}dt \\ &= -\frac{1}{i\omega}e^{-i\omega t}\Big|_{t=0}^{t=\frac{1}{2}} + \frac{1}{i\omega}e^{-i\omega t}\Big|_{t=\frac{1}{2}}^{t=1} \\ &= \frac{1}{i\omega}\left[1 - e^{-i\omega/2} + e^{-i\omega} - e^{-i\omega/2}\right] \\ &= \frac{1}{i\omega}\left(1 - e^{-i\omega/2}\right)^2 \end{aligned}$$

$$= \frac{1}{i\omega}\left(e^{i\omega/4} - e^{-i\omega/4}\right)^2 e^{-i\omega/2}$$
$$= \frac{4i}{\omega} e^{-i\omega/2} \sin^2\left(\frac{\omega}{4}\right).$$

It might worry some that this is explicitly a complex valued function. It should not. The mother wavelet $\psi(t)$ is neither even nor odd, therefore it will have a complex valued Fourier transform. However it is true that for practical applications complex valued functions of the wave frequency ω are a nuisance. Either we work with Fourier sine or cosine transforms, or, more usually, symmetric functions that represent practical aspects of the time series being represented by wavelets are used. These are the auto-covariance and autocorrelation functions mentioned at the beginning of the chapter. The auto-covariance of a given function (time series) $f(t)$ is defined as follows:

$$R_f(\tau) = \int_{-\infty}^{\infty} f(t) f(t-\tau) dt.$$

The only difference between this and the autocorrelation is normalisation. The autocorrelation is the normalised auto-covariance ($= R_f(\tau)/|R_f(\tau)|$). Let us spend a little time with this auto-covariance function. The mathematical definition above is straightforward enough, and we shall calculate it soon. Practically, it is a measure of how much a signal $f(t)$ agrees with itself (or *correlates* with itself) τ seconds later. It is a maximum at $\tau = 0$ as this agreement is exact then. It is also a symmetric function. This is easy to show:

$$R_f(-\tau) = \int_{-\infty}^{\infty} f(t) f(t+\tau) dt$$

changing the variable through $u = t + \tau$ gives $dt = du$ and $t = u - \tau$ with the limits staying at $-\infty$ and ∞, so

$$R_f(-\tau) = \int_{-\infty}^{\infty} f(u-\tau) f(u) du = R_f(\tau)$$

from the definition, and the result is proved trivially. It is also the case that the Fourier transform of the auto-variance has a practical meaning. We write

$$S_f(\omega) = \int_{-\infty}^{\infty} R_f(\tau) e^{-i\omega\tau} d\tau$$

and call $S_f(\omega)$ the *spectral density* (or sometimes the *power spectral density*). The spectral density gives the distribution of energy through the density spectrum of the time series $f(t)$. So a single frequency or sine wave would be a spike (Dirac-δ function), whilst if $S_f(\omega) =$ constant this would mean all frequencies have equal energy in the signal, which electronic engineers call *white noise*. However this takes us outside the scope of an undergraduate mathematics text and firmly into signal processing. Let us retreat and do some calculation.

Example 7.4 *Determine the auto-covariance of the mother wavelet function, and hence evaluate its Fourier transform.*

Solution It is not wise to plunge straight into calculation here. The mother wavelet $\psi(t)$ is given by:

$$\psi(t) = \begin{cases} 1, & 0 \le t < \frac{1}{2} \\ -1, & \frac{1}{2} \le t < 1 \\ 0, & \text{otherwise} \end{cases}$$

and so is only non-zero inside the interval $[0, 1]$. This means that the function $\psi(t-\tau)$ is also zero outside the different interval $[\tau, 1+\tau]$. Thus should τ be less than -1 or greater than 1 the product $\psi(t)\psi(t-\tau)$ will be identically zero because the two intervals where they are non zero fail to overlap, which implies that $R_\psi(\tau)$ will also be zero outside the range $[-1, 1]$. Inside this range, the value of $R_\psi(\tau)$ is calculated piecemeal. It will be done explicitly, even though symmetry ($R_\psi(\tau)$ is an even function) could be invoked. It is better to use this as a check as the algebra can be messy. For τ in the interval $[-1, -1/2]$ we have

$$R_\psi(\tau) = \int_{-\infty}^{\infty} \psi(t)\psi(t-\tau)dt = \int_0^{1+\tau} (-1)(1)dt = -1-\tau$$

since the overlap region in t space is from 0 to $1+\tau$ which of course is between 0 and $1/2$ for this range of τ. The next range of τ is $[-1/2, 0]$ which gives:

$$\begin{aligned} R_\psi(\tau) &= \int_0^{(1/2)+\tau} (1)(1)dt + \int_{(1/2)+\tau}^{1/2} (-1)(1)dt + \int_{1/2}^{1+\tau} (-1)(-1)dt \\ &= 1/2+\tau-(1/2-1/2-\tau)+(1+\tau-1/2) = 3\tau+1. \end{aligned}$$

The third range for τ is $[0, 1/2]$ and the calculation for $R_\psi(\tau)$ is:

$$\begin{aligned} R_\psi(\tau) &= \int_\tau^{1/2} (1)(1)dt + \int_{1/2}^{(1/2)+\tau} (-1)(1)dt + \int_{(1/2)+\tau}^{1} (-1)(-1)dt \\ &= 1/2-\tau-(\tau+1/2-1/2)+(1-\tau-1/2) = 1-3\tau. \end{aligned}$$

Finally, the fourth and last non-zero range for τ is $[1/2, 1]$ so the only non-zero interval $[1/2, \tau]$ giving

$$R_\psi(\tau) = \int_{-\infty}^{\infty} \psi(t)\psi(t-\tau)dt = \int_\tau^1 (-1)(1)dt = \tau-1.$$

Thus written in terms of τ the function $R_\psi(\tau)$ is

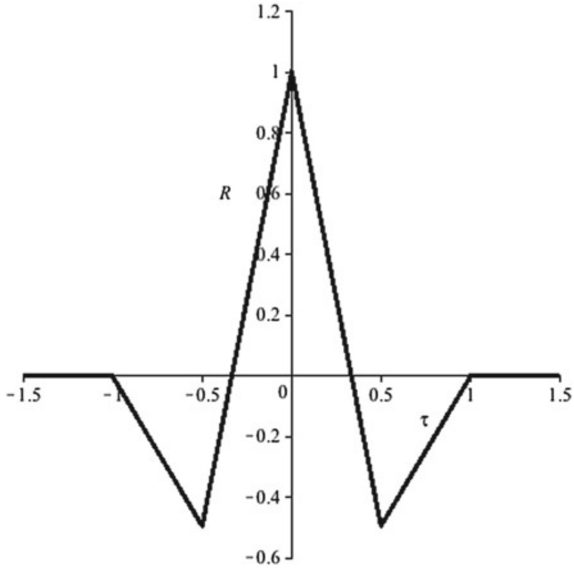


Fig. 7.5 The auto-covariance function for the mother wavelet $R_\psi(\tau)$

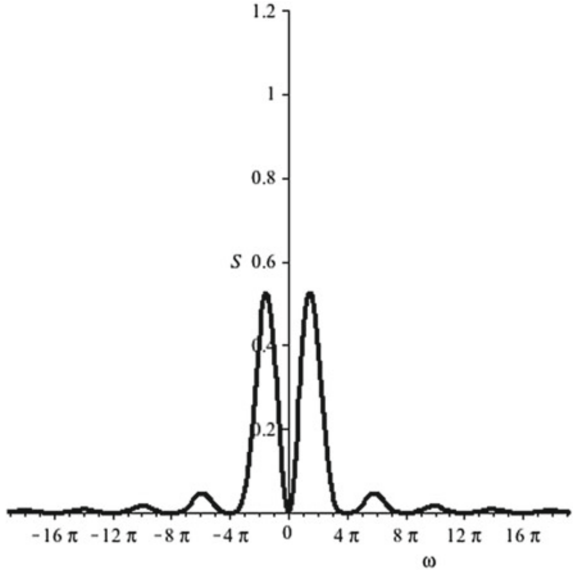


Fig. 7.6 The spectral density function for the mother wavelet $S_\psi(\omega)$

$$R_\psi(\tau) = \begin{cases} -1-\tau, & -1 \le \tau \le -\frac{1}{2} \\ 1+3\tau, & -\frac{1}{2} < \tau \le 0 \\ 1-3\tau, & 0 < \tau \le \frac{1}{2} \\ \tau-1, & \frac{1}{2} < \tau \le 1. \end{cases}$$

This auto-covariance function is shown in Fig. 7.5; it is indeed as expected, an even function. Now let us turn our attention to finding its Fourier transform. So, we evaluate

$$S_\psi = \int_{-\infty}^{\infty} R_\psi(\tau) e^{-i\omega\tau} d\tau$$

which using the definition of $R_\psi(\tau)$ just derived leads to

$$S_\psi = \int_0^{1/2} 2(1-3\tau)\cos(\omega\tau)d\tau + \int_{1/2}^{1} 2(\tau-1)\cos(\omega\tau)d\tau$$

once the symmetry of $R_\psi(\tau)$ is utilised. S_ψ is now real and can be evaluated either by hand using integration by parts or by using software. The resulting expression is:

$$S_\psi = \frac{4}{\omega^2}\left(\cos^2\left(\frac{1}{2}\omega\right) - 2\cos\left(\frac{1}{2}\omega\right) + 1\right)$$

and this function is displayed in Fig. 7.6. It is symmetric, which is obvious from its definition, but in practical terms as ω is a frequency it is positive and only the part that corresponds to positive values of ω is considered. $S = 0$ when $\omega = 0$ which means that there is no steady time independent part of the signal. This is certainly true for all wavelets, because no wavelet persists outside a unit window. It is also true that $S \to 0$ as $\omega \to \infty$. This implies that there is also no energy in the very short time scales (high frequency). Although less obvious, it can also be deduced from the mother wavelet that peak energy occurs around $t = 1$ corresponding to $\omega = (2 + 4N)\pi$, N an integer. In general, it is possible to look at the auto-covariance and spectral density of a time series and deduce important information about the parent signal in terms of dominant time scales and frequencies. Through this knowledge it is possible to model the effects that caused the time series. This might be seismic effects (e.g. earthquakes), engineering effects (e.g. vibrations or ocean waves) or medical signals (e.g. heart rates or brain waves). This is the study of signal processing and those interested in this are steered towards specialist texts such as Mallat's book.

7.6 Wavelets and Fourier Series

Before proceeding with some wavelet specific results, let us look at their relation with Fourier series. A feature of most time series (let's call a typical one $f(t)$) is that they start at $-\infty$ and go on forever (to ∞); moreover they do not repeat or even stay at zero for any stretch of time. This is dealt with by truncation, but another idea would be not only to truncate the series to lie between 0 and 2π say, but instead of having $f(t) = 0$ outside this range simply turn the series into a periodic function $f_p(t)$ so that

$$f_p(t) = \sum_{n=-\infty}^{\infty} f(t + 2\pi n).$$

Defined above, $f_p(t)$ is certainly periodic and so possesses a Fourier series. Let's seek a complex Fourier series as outlined in Sect. 4.4.:

$$f_p(t) = \sum_{-\infty}^{\infty} c_k e^{ikt}$$

with the coefficients c_k given by

$$\begin{aligned}
c_k &= \frac{1}{2\pi} \int_0^{2\pi} f_p(t) e^{-ikt} dt \\
&= \frac{1}{2\pi} \int_0^{2\pi} \sum_{n=-\infty}^{\infty} f(t + 2\pi n) e^{-ikt} dt \\
&= \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \int_0^{2\pi} f(t + 2\pi n) e^{-ikt} dt \\
&= \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \int_{2\pi n}^{2\pi(n+1)} f(\xi) e^{-ik(\xi - 2\pi n)} d\xi
\end{aligned}$$

where the change of variable $\xi = t + 2\pi n$ has been used. As n starts at $-\infty$ and runs all the way through to ∞ the right hand side sums term by term to the integral

$$c_k = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\xi) e^{-ikt} d\xi = \frac{1}{2\pi} \hat{f}(k)$$

provided of course that the sum exists. We can prove that the improper integral converges as follows: we are are looking at only a portion of the time series $f(t)$ and repeating it every 2π. Hence in the integrand $f(t)$ is multiplied by trigonometric functions (e^{-ikt}) so by the Riemann Lebesgue lemma (Theorem 4.2) the integral will converge. Here $\hat{f}(k)$ is, by definition, the Fourier transform of $f(t)$. Back in Chap. 4 k would have been thought to be an integer, however in terms of Chap. 6, k is a variable. If the above is slightly generalised from $f(t)$ having period 2π to having period T then we have

$$f_p(t) = \sum_{n=-\infty}^{\infty} f(t + nT) = \frac{1}{T} \sum_{k=-\infty}^{\infty} \hat{f}(k\omega_0) e^{ik\omega_0 t}.$$

This last equality is a special case of Poisson's sum formula. It is usually generalised through writing f(at) instead of f(t) but $T = 2\pi$ once again, whence

$$\sum_{n=-\infty}^{\infty} f(t + 2\pi an) = \frac{1}{2\pi a} \sum_{k=-\infty}^{\infty} \hat{f}\left(\frac{k}{a}\right) e^{ikt/a}. \tag{7.3}$$

Poisson's sum formula is simply a special Fourier series, but the Fourier coefficients of this special complex Fourier series happen to be the Fourier transform of $f(t)$ the original full time series itself. Of course, the whole point originally was that the Fourier transform of the time series $f(t)$ was hard to find. However it is not difficult to derive the following two formulas. They are actually easily obtained from Eq. (7.3) by changing variables. Equations (7.3) and (7.4) (or (7.3) and (7.5)) look like finite Fourier transforms and their inverse.

$$\sum_{k=-\infty}^{\infty} \hat{f}(\omega + 2\pi k) = \sum_{k=-\infty}^{\infty} f(k) e^{-ik\omega} \tag{7.4}$$

and

$$\sum_{k=-\infty}^{\infty} \hat{f}\left(\frac{\omega + 2\pi k}{a}\right) = \sum_{k=-\infty}^{\infty} f(ak) e^{-ik\omega}. \tag{7.5}$$

If $a = 1/(2\pi)$ in Eq. (7.3) it becomes the beautifully simple:

$$\sum_{n=-\infty}^{\infty} f(t + n) = \sum_{k=-\infty}^{\infty} \hat{f}(2\pi k) e^{2\pi ikt}. \tag{7.6}$$

Consider the special case of the Fourier transform of $f(t)$ that gives

$$\hat{f}(2\pi k) = \delta_{0,k} \text{ where } k \in \mathbb{Z}.$$

So if 0 is excluded, the Kronecker delta is always zero which means that the sum on the right of Eq. (7.6) only has one non-zero term, leading to

$$\sum_{n=-\infty}^{\infty} f(t + n) = 1.$$

The left hand side thus sums to one and is called a Partition of Unity. Functions that when all added up equal one are (perhaps surprisingly) very useful. An example of one is the Haar function:

$$B_1(t) = \chi_{[0,1)}(t) = \begin{cases} 1, & 0 \le t < 1 \\ 0, & \text{otherwise} \end{cases}$$

introduced earlier. The Fourier transform of this is:

$$\hat{B}_1(\omega) = \int_0^1 e^{-i\omega t} dt = \frac{1 - e^{-i\omega}}{i\omega}$$

so that

$$\hat{B}_1(0) = \lim_{\omega \to 0} \frac{1 - e^{-i\omega}}{i\omega} = 1$$

and also

$$\hat{B}_1(2\pi k) = 0, \; k = \pm 1, \pm 2 \ldots$$

hence

$$\sum_{n=-\infty}^{\infty} B_1(t + n) \equiv 1.$$

The function $B_1(t)$ is called a first order B-Spline function.

Example 7.5 *Calculate $B_2(t) = B_1(t) * B_1(t)$. Use the convolution theorem to find its Fourier transform and hence generalise the result.*

Solution Using direct integration, if $0 \le t \le 1$ we have

$$B_2(t) = B_1 * B_1 = \int_0^t B_1(\tau) B_1(t - \tau) d\tau = \int_0^t d\tau = t$$

whereas if $0 \le t - \tau \le 1$ (which is $t - 1 \le \tau \le t$) we have

$$B_2(t) = B_1 * B_1 = \int_0^t B_1(\tau) B_1(t - \tau) d\tau = \int_{t-1}^1 d\tau = 2 - t.$$

So B_2 is a triangular pulse:

$$B_2(t) = \begin{cases} t, & 0 \le t \le 1 \\ 2 - t, & 1 \le t \le 2 \\ 0, & \text{otherwise.} \end{cases}$$

The Fourier transform can be easily found from the convolution theorem applied to Fourier transforms, namely that the Fourier transform of a convolution is the product of the two component Fourier transforms, so

$$\hat{B}_2 = [\hat{B}_1(\omega)]^2 = \left(\frac{1 - e^{-i\omega}}{i\omega} \right)^2.$$

$B_2(t)$ is called the second order B-spline. The nth order B-spline is found recursively through the convolution

$$B_n(t) = B_{n-1}(t) * B_1(t)$$

hence, taking Fourier transforms and successively putting $n = 2, 3$ and so on gives, by induction,

$$\hat{B}_n(\omega) = \left(\frac{1 - e^{-i\omega}}{i\omega}\right)^n.$$

The definition of $B_1(t)$ (it is the top hat function 1 between 0 and 1, zero elsewhere) enables us to compute the convolution explicitly giving the simple integral

$$B_n(t) = \int_0^1 B_{n-1}(t - \tau)d\tau.$$

The principal applications of wavelets is in signal processing, but this is not a textbook on signal processing. However, sampling a time series is so central that it is useful to have what may seem a short digression here, but it really is not as what follows is securely applied mathematics. Suppose there is a time series $f(t)$ that needs to be analysed. To do so it is normal to sample it at some fixed period, call this period h. Suppose also that attention is limited to analysing a specific range of frequencies (called the *bandwidth*). Call this range of frequencies 2Ω. If $h = \pi/\Omega$ then this is the largest value that h can have and still hope to recover the important features of the original signals, it is called the Nyquist frequency. All this being said, let us apply some of the previous mathematics to sampling a time series. First of all, sampling is done on the wave frequency version of the signal which we shall assume is the Fourier transform $\hat{f}(\omega)$ which is given by

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt.$$

Suppose that this integral can be approximated using say Simpson's rule through:

$$\hat{F}(\omega) = h\sum_{k=-\infty}^{\infty} f(kh)e^{-ik\omega h} \approx \hat{f}(\omega).$$

We can now use a version of the Poisson sum (Eq. 7.5) to write this as:

$$\hat{F}(\omega) = \sum_{k=-\infty}^{\infty} \hat{f}\left(\frac{\omega h + 2\pi k}{h}\right) = \hat{f}(\omega) + \sum_{k=-\infty}^{-1} \hat{f}\left(\omega + \frac{2\pi k}{h}\right) + \sum_{k=1}^{\infty} \hat{f}\left(\omega + \frac{2\pi k}{h}\right).$$

This last equation tells us that the approximation $\hat{F}(\omega)$ consists of $\hat{f}(\omega)$ itself plus infinitely many copies of $\hat{f}(\omega)$ shifted by $2\pi/h$ along the ω-axis. So the question arises how do we retrieve the original time series having only access to the approximate frequency distribution. Suppose then that we just have $\hat{F}(\omega)$ in this form, the way to proceed is to isolate the original $\hat{f}(\omega)$ by windowing out the shadows through a window function:

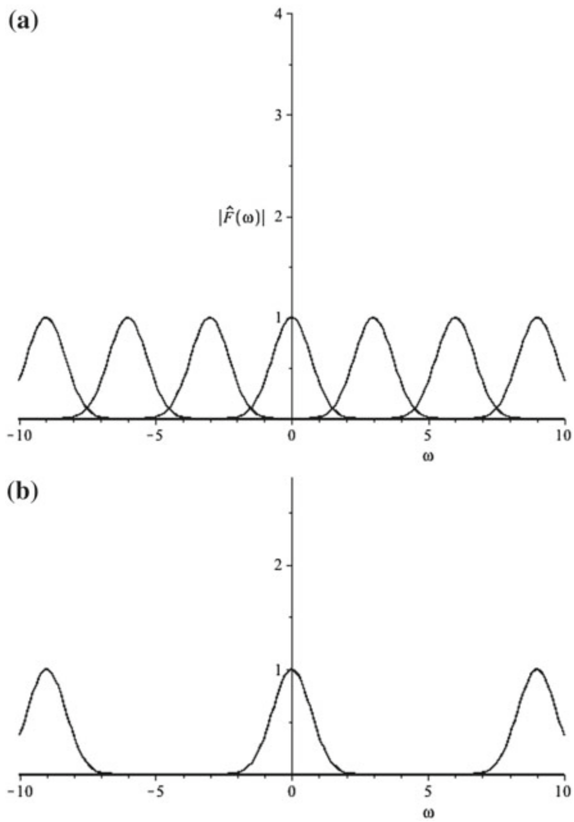


Fig. 7.7 **a** (*above*) not enough sampling **b** (*below*) too much sampling. The vertical axis on both graphs are $|\hat{F}(\omega)|$

$$\hat{W}(\omega) = \begin{cases} 1, & |\omega| < \Omega \\ 0, & \text{otherwise.} \end{cases}$$

Of course we must be sure that the images of $\hat{f}(\omega)$ do not overlap. It is also prudent not to over sample, this causes large gaps between the shadow image which in itself does not seem a problem, but can be expensive (why get more data than the minimum required?). Both over and under sampled time series frequencies are shown in Fig. 7.7. The other essential fact to enable the original series to be retrieved from the Fourier transform $\hat{f}(\omega)$ is to ensure band limitation, and this has been done by defining the bandwidth as 2Ω. The shape of $\hat{f}(\omega)$ can be anything of course, it has a Gaussian shape in Fig. 7.7 merely as an illustration. To ensure there is only one image of $\hat{f}(\omega)$ in the version we have, namely $\hat{F}(\omega)$, we multiply it by a window so that all the repeats are excluded so:

$$\hat{f}(\omega) = \hat{F}(\omega)\hat{W}(\omega).$$

Inverting this we have immediately

$$f(t) = F(t) * W(t)$$

where $\hat{W}(\omega)$ is defined above. From the beginning of the chapter, it's inverse Fourier transform $W(t)$ is the sinc function:

$$W(t) = \frac{\sin(\Omega t)}{\pi t}. \tag{7.7}$$

We also use a result from Chap. 6 (see Exercise 6.14), namely:

$$\delta(t - kh) = 2\pi \int_{-\infty}^{\infty} e^{i\omega(t-kh)} d\omega.$$

We now derive a useful expression, starting from the inverse Fourier transform of $F(t)$, the approximation to the original time series $f(t)$ obtained through the inverse Fourier transform of the Simpson's rule summation of the exact inverse Fourier transform of $f(t)$ (hope you're keeping up). We have

$$F(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{F}(\omega) e^{i\omega t} d\omega = \frac{h}{2\pi} \int_{-\infty}^{\infty} \sum_{k=-\infty}^{\infty} f(kh) e^{-ikh\omega} e^{i\omega t} d\omega.$$

Manipulation of the right hand side gives

$$\begin{aligned} F(t) &= \frac{h}{2\pi} \sum_{k=-\infty}^{\infty} f(kh) \int_{-\infty}^{\infty} e^{i\omega(t-kh)} d\omega \\ &= h \sum_{k=-\infty}^{\infty} f(kh)\delta(t - kh) \qquad \text{from the result above, see Exercise 6.14.} \end{aligned}$$

We now turn our attention to $f(t)$ itself and use the result just obtained on the convolution $f(t) = F(t) * W(t)$ to give

$$\begin{aligned} f(t) &= h \sum_{k=-\infty}^{\infty} f(kh) \int_{-\infty}^{\infty} \delta(\tau - kh) W(t - \tau) d\tau \\ &= h \sum_{k=-\infty}^{\infty} f(kh) W(t - kh) \qquad \text{using a property of the } \delta \text{ function} \\ &= h \sum_{k=-\infty}^{\infty} f(kh) \frac{\sin[\Omega(t - kh)]}{\pi(t - kh)} \qquad \text{from Eq. (7.7)} \end{aligned}$$

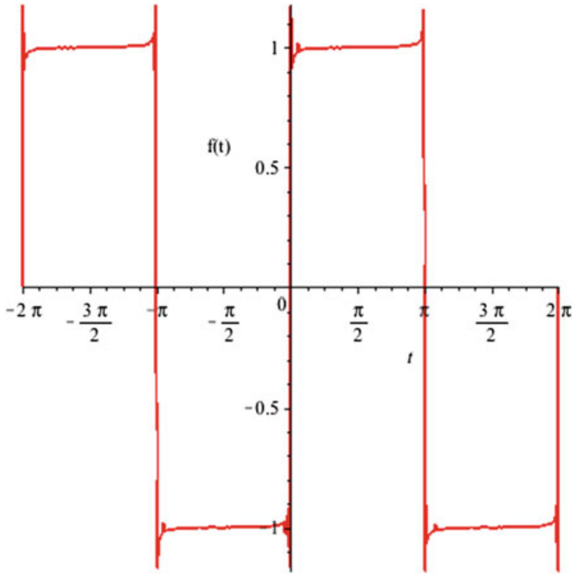


Fig. 7.8 The square wave. The first 100 terms of the Fourier series $f(t) = \frac{4}{\pi}\sum_{n=1}^{100}\frac{\sin\{(2n-1)t\}}{2n-1}$, and Gibbs phenomena remains clear

$$= \sum_{k=-\infty}^{\infty} f(kh)\frac{\sin(\Omega t - k\pi)}{\Omega t - k\pi} \qquad \text{using } h = \pi/\Omega.$$

This result is interesting. It tells us that $f(t)$ is obtained *exactly* by the sum on the right of the last equation. Not really a surprise as at the points where $t = kh$ the quotient $\sin[\Omega(t-kh)]/[\Omega(t-kh)]$ is exactly unity and it is zero at all other values of k where $t \neq kh$. Hence this is an interpolation formula for $f(t)$, reconstructed via the sinc function. Before returning to the central theme of this chapter, wavelets, let us apply similar analysis to say a bit more about Gibbs phenomenon and aliasing which were briefly mentioned in Chap. 4 at the end of Sect. 4.4. Both of these arise because a Fourier series is being truncated, but they can be quite pronounced even if plenty of terms of the Fourier series are being summed. Gibbs phenomenon is a particular nuisance when trying to represent square waves with Fourier series, see Fig. 7.8, square waves have an abrupt discontinuity where the value of the series jumps and at these jumps the truncated series overshoot and oscillates.

Before doing any mathematics it is useful to describe in words what both aliasing and Gibbs phenomenon are. First of all aliasing. In the upper diagram of Fig. 7.7, the front of one Gaussian image overlaps with the rear of the next one. This is aliasing. It occurs in signal processing because insufficient samples are being taken. Gibbs phenomenon is actually more interesting mathematically. It is the overshoot and subsequent oscillation of the approximation to a periodic function by a truncated Fourier series. Gibbs phenomenon is worst at discontinuities of the original function. Let us derive what happens when we truncate a Fourier series. We keep to the complex form and define a truncated Fourier series as follows:

$$f_N(t) = \sum_{k=-N}^{N} c_k e^{ik\omega_0 t}$$

where $f(t)$ has period T and N is an integer (usually large). The co-efficients c_k are given by

$$c_k = \frac{1}{T}\int_{-T/2}^{T/2} f(t)e^{-i\omega_0 t}dt.$$

All this was covered in Chap. 4. We substitute for c_k back into the truncated Fourier series to get

$$f_N(t) = \sum_{k=-N}^{N} \frac{1}{T}\int_{-T/2}^{T/2} f(\tau)e^{-ik\omega_0\tau}e^{ik\omega_0 t}d\tau$$

where the dummy variable τ is necessary as t is already in use. Changing the order of summation and integration gives

$$\begin{aligned} f_N(t) &= \frac{1}{T}\int_{-T/2}^{T/2} f(\tau)\sum_{k=-N}^{N} e^{ik\omega_0(t-\tau)}d\tau \\ &= \frac{1}{T}\int_{-T/2}^{T/2} f(\tau)\frac{\sin\left(\left(N+\frac{1}{2}\right)(t-\tau)\,\omega_0\right)}{\sin\left(\frac{1}{2}(t-\tau)\,\omega_0\right)}d\tau. \end{aligned}$$

The summation of the complex exponential series from $-N$ to N is an exercise in summing a geometric progression (write it as $e^{-iNp} \times (1 + e^{ip} + \cdots + e^{2iNp})$ where $p = (t-\tau)\omega_0$). What we have derived is an integral for the truncated function (or signal) in terms of the real one but weighted by a function of the general form $\sin(nt)/\sin(\frac{1}{2}t)$. This function is shown for $n = 25$ in Fig. 7.9. It has the value $\pm 2n$ at $t = 2\pi k$, $k \in \mathbb{Z}$ with a positive value at the origin and alternating every $t = 2\pi$. As $n \to \infty$ this function resembles a series of positive and negative Dirac δ-functions. A Shah or comb function but with teeth, width 2π apart, in alternating directions.

It is the function $\dfrac{\sin\left(\left(N+\frac{1}{2}\right)(t-\tau)\,\omega_0\right)}{\sin\left(\frac{1}{2}(t-\tau)\,\omega_0\right)}$ that is responsible for the spurious oscillations around discontinuities. No matter how large N is, if $f(t)$ is discontinuous at some value then this function will oscillate. In fact the larger N the larger the number of oscillations. However the larger N also the nearer the whole integral is to the correct value of $f(t)$. At values of t where $f(t)$ is continuous this function will then ensure $f_N(t)$ behaves like $f(t)$. Very close to discontinuities of $f(t)$ these discontinuities will always be visible and the larger N the more rapid the oscillation. This explains why taking very large values of N does not cure Gibbs phenomenon, functions like this ratio of sinusoids simply do not act as very good interpolation functions across jump discontinuities. For the rigorous proofs and more examples you are referred to the excellent text by Pinkus and Zafrany (1997) (see also Chap. 14 of the classic text Jeffries and Jeffries (1956)).

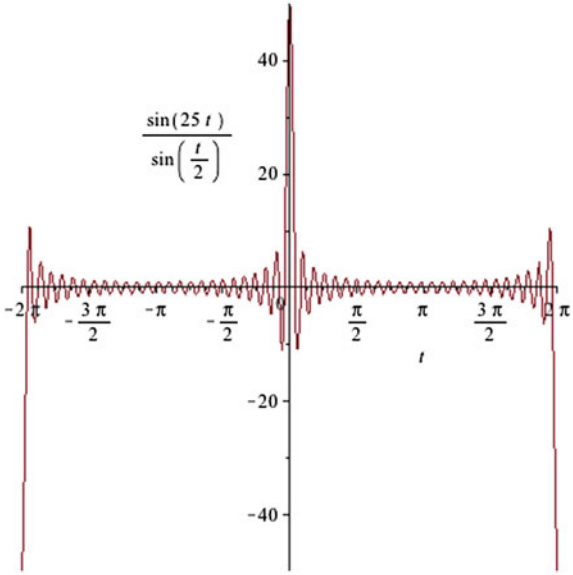


Fig. 7.9 The function $\sin(25t)/\sin(t/2)$ shows typical behaviour

7.7 Localisation

What we really haven't done yet is to exploit the essential feature of wavelets, their limited time range. The real problem is that although Haar wavelets and the generations defined earlier in this chapter are zero outside the range [0, 1], their Fourier transforms are not. They are concentrated and decay to zero for large ω, but they are not identically zero for finite values of ω. In practical applications this is not acceptable, however it is not possible to keep both the time and corresponding frequency distributions confined to strict windows simultaneously. This can be proved. Looking at the mathematics, the reason is clearly the presence of $e^{-i\omega t}$ in the integrand of the Fourier integral. Although the integrals in the Fourier transforms are convergent, they converge slowly and they oscillate while doing so. This is not a desirable feature for a wavelet representation of a signal. So we do the best we can. It turns out that the best possible in terms of putting the whole of both the time function and its Fourier transform inside a box in $t - \omega$ space is that both $f(t)$ and its transform $\hat{f}(\omega)$ are Gaussian. This is shown in Fig. 7.10. Recall from Exercise 6.2 that the Fourier transform of $e^{-\frac{1}{2}t^2}$ is the almost identical function $\sqrt{\pi}e^{-\frac{1}{2}\omega^2}$ in wave space. So the Gaussian function seems to achieve this windowing effect most effectively. The two Gaussian functions are displayed on each axis and most of the points where both t and ω values non-zero lie inside this box. Not all of them of course as the Gaussian functions only tend to the axes asymptotically. This is hard to see in the diagram. Also more trivially in this figure the two Gaussian functions are centred at $t = 5, \omega = 5$ and both t and ω have been scaled by $1/\sqrt{2}$ for convenience.

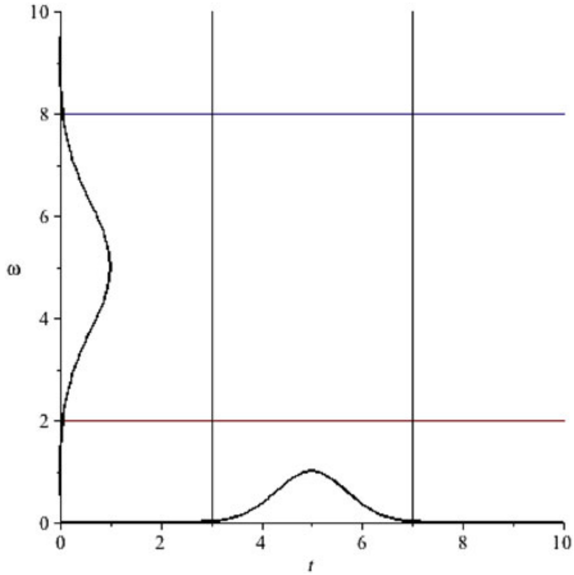


Fig. 7.10 The Heisenberg box where the most of the information lies in $\omega - t$ space

In order to be quantitative we need to define some terms. First of all define the window function $\chi_{[-\tau,\tau)}$. This is a top hat function lying between $-\tau$ and τ that includes the value $-\tau$ but not the value τ. An example will be done using this window function, however for a general function of time $p(t)$, the following quantity is defined:

$$\mu_p = \frac{1}{||p(t)||^2} \int_{-\infty}^{\infty} t|p(t)|^2 dt$$

where $||p(t)||$ is the norm defined here as the positive square root of

$$||p(t)||^2 = \int_{-\infty}^{\infty} |p(t)|^2 dt.$$

This is called the *centre* of the function $p(t)$. It is the mean of the absolute value squared and often equals zero but in general is not. The root-mean-square (RMS) Δ_p is a measure of spread (like variance or standard deviation) and is defined by

$$\Delta_p = \frac{1}{||p(t)||} \left[\int_{-\infty}^{\infty} (t-\mu)^2 |p(t)|^2 dt \right]^{1/2}.$$

These two quantities are used for analysis of the representation of sampled time series by wavelets. Let us do a computational example.

Example 7.6 *Calculate the centre and RMS value for the general window function* $\chi_{[-\tau,\tau)}$ *defined by*

$$\chi_{[-\tau,\tau)} = \begin{cases} 1, & -\tau \le t < \tau \\ 0, & otherwise. \end{cases}$$

Find also the centre and RMS values of its Fourier transform.

Solution The norm of $\chi_{[-\tau,\tau)}$ is given by the square root of the integral

$$\int_{-\infty}^{\infty} |\chi_{[-\tau,\tau)}|^2 dt = \int_{-\tau}^{\tau} dt = 2\tau.$$

So $||\chi_{[-\tau,\tau)}|| = \sqrt{2\tau}$. The centre, μ_χ is thus given by:

$$\mu_\chi = \frac{1}{||\chi_{[-\tau,\tau)}||^2} \int_{-\infty}^{\infty} t|\chi_{[-\tau,\tau)}|^2 dt = \frac{1}{2\tau} \int_{-\tau}^{\tau} t dt = 0.$$

The RMS value of μ_χ is

$$\Delta_\chi = \frac{1}{||\chi_{[-\tau,\tau)}||} \left[\int_{-\infty}^{\infty} (t - \mu_\chi)^2 |\chi_{[-\tau,\tau)}|^2 dt \right]^{1/2} = \frac{1}{\sqrt{2\tau}} \left[\int_{-\tau}^{\tau} t^2 dt \right]^{1/2} = \frac{1}{\sqrt{2\tau}} \sqrt{\frac{2}{3}\tau^3}$$

which simplifies to

$$\frac{\tau}{\sqrt{3}}.$$

The Fourier transform of $\chi_{[-\tau,\tau)}$ is

$$\hat{\chi}(\omega) = \int_{-\infty}^{\infty} \chi_{[-\tau,\tau)}(t) e^{-i\omega t} dt = \int_{-\tau}^{\tau} e^{-i\omega t} dt = \left[\frac{e^{-i\omega t}}{-i\omega} \right]_{-\tau}^{\tau}$$

which simplifies to

$$\hat{\chi}(\omega) = \frac{2\sin(\omega\tau)}{\omega}.$$

The norm in frequency space is $||\hat{\chi}(\omega)||$ where

$$||\hat{\chi}(\omega)||^2 = \int_{-\infty}^{\infty} 4 \left| \frac{\sin^2 \omega t}{\omega^2} \right| d\omega.$$

Using the standard integral

$$\int_0^{\infty} \frac{\sin^2(x)}{x^2} dx = \frac{\pi}{2}$$

we see that $||\hat{\chi}(\omega)|| = 2\sqrt{\pi\tau}$. So, using the definition of centre,

$$\mu_{\hat{\chi}} = \frac{1}{||\hat{\chi}(\omega)||^2} \int_{-\infty}^{\infty} \omega \left| \frac{4\sin^2 \omega t}{\omega^2} \right| d\omega.$$

Since the integrand is an odd function, and it is convergent, its value must be zero. So $\mu_{\hat{\chi}} = 0$. Using the definition of RMS value we have

$$\Delta_{\hat{\chi}} = \frac{1}{||\hat{\chi}(\omega)||}\left[\int_{-\infty}^{\infty}\omega^2\left|\frac{2\sin\omega t}{\omega}\right|^2 d\omega\right]^{1/2}$$

and since the integrand oscillates between 0 and 4 the improper integral is not convergent. Thus in simple terms, $\Delta_{\hat{\chi}} = \infty$.

In the above example, $\Delta_\chi \Delta_{\hat{\chi}}$ is infinite. If we move away from this specific example, in general, let us label the RMS in t space Δ_f and the RMS in ω space $\Delta_{\hat{f}}$. The product of these two quantities $\Delta_f \Delta_{\hat{f}}$ has to obey the inequality

$$\Delta_f \Delta_{\hat{f}} \geq \frac{1}{2}.$$

The inequality becomes an equality when $f(t)$ is Gaussian (see Exercise 7.5). The relation itself resembles Heisenberg's uncertainty principle which will be familiar to those who have studied quantum mechanics. It has similar implications. For example if one of Δ_f or $\Delta_{\hat{f}}$ is zero then the other has to be infinite. So if there is no uncertainty at all in the value of say the sampled time series $f(t)$ then the Fourier transform of the sampled signal cannot be found. If both are finite, then the above inequality has to hold. These RMS values indicate the variability of the signal and its Fourier transform, so that the product can never be zero means that there has to be uncertainty in Δ_f (the signal or its sampled version) and $\Delta_{\hat{f}}$ (the transform), and these uncertainties are minimised if the product is $1/2$. As we said above, this minimisation is attained if the signal is Gaussian; in Fig. 7.10 it might look like the Gaussian pulses are completely confined however the Gaussian distributions asymptote to both axes, so there are always points outside the Heisenberg box where $f(t)$ and $\hat{f}(\omega)$ are non-zero. For distributions that are composed from wavelets, the transform will oscillate, but these RMS values will not as they are constructed to be positive. Unfortunately the integrals are seldom analytically tractable. This next example involves a very straightforward function, yet one integral in particular is very hard to evaluate.

Example 7.7 *Determine the Fourier transform of the function*

$$f(t) = \begin{cases} -1, & -1 < t < -\frac{1}{2} \\ 1, & \frac{1}{2} < t < 1 \\ 0, & \textit{otherwise}. \end{cases}$$

Hence find the values of $\mu_t, \Delta_t, \mu_\omega, \Delta_\omega$.

Solution The Fourier transform is straightforward enough to find:

$$
\begin{aligned}
\hat{f}(\omega) &= \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt \\
&= \int_{-1}^{-\frac{1}{2}} (-1)e^{-i\omega t}dt + \int_{\frac{1}{2}}^{1} e^{-i\omega t}dt \\
&= 2\int_{\frac{1}{2}}^{1} \cos(\omega t)dt \\
&= \frac{2}{\omega}\left[\sin\omega - \sin(\frac{1}{2}\omega)\right].
\end{aligned}
$$

Next we need to find the two norms. $||f(t)|| = 1$ without problems, however

$$
||\hat{f}(\omega)|| = 16\int_0^{\infty} \frac{1}{\omega^2}\left|\sin\omega - \sin\frac{1}{2}\omega\right|^2 d\omega
$$

and this integral is non trivial to compute. MAPLE gives the answer π. For those who want to try to compute the integral by hand, use $\sin\omega - \sin\frac{1}{2}\omega = 2\sin\frac{1}{4}\omega\cos\frac{3}{4}\omega$ then rearrange the integrand so that it can be put in terms of known integrals such as

$$
\int_0^{\infty} \frac{\sin^2 x}{x^2}dx = \frac{\pi}{2}.
$$

The more ambitious can try more advanced methods (see the next chapter; use residue calculus, also see Weinberger (1965)). Now we can find the rest of the required quantities $\mu_f(t) = 0$ by asymmetry,

$$
\Delta_f(t) = \left[\int_{-\infty}^{\infty} t^2|f(t)|^2 dt\right]^{1/2} = \sqrt{\frac{7}{12}}
$$

following routine integration. Similarly $\hat{f}(\omega) = 0$ also by asymmetry, and

$$
\Delta_{\hat{f}} = \left[\frac{1}{\pi^2}\int_{-\infty}^{\infty}\left(\sin\omega - \sin\frac{1}{2}\omega\right)^2 d\omega\right]^{\frac{1}{2}}
$$

which is clearly infinite since the integrand remains finite for large ω so the integral fails to converge. This also means that, as before $\Delta_f\Delta_{\hat{f}} = \infty$.

7.8 Short Time Fourier Transform

The next logical step is to define the Short-Time Fourier transform. This is done as follows. First of all the function of t, $f(t)$, is windowed by multiplying it by a suitable time limited function $b(t)$. We avoid calling the window function w to avoid confusion with frequency ω. One such window function could be the top hat function, say the wavelet $\chi_{[-\tau,\tau)}$. Whatever choice, this defines the windowed version of the original function $f_b(t)$

$$f_b(t) = f(t)b(t).$$

The Fourier transform of $f_b(t)$ is called the Short-Time Fourier transform abbreviated STFT. To find the STFT we simply find the Fourier transform of this product and make use of convolution. In practical use, the Gaussian shape or something similar is used in place of the wavelet $\chi_{[-\tau,\tau)}$. The application of this concept to the analysis of time series will be left to specialist texts, however there are interesting generalities worth including here. Suppose that the window is centred on a time t_0 so that the windowed function is $f(t)b(t-t_0)$. In general, the window function could have complex values, but let us keep it simple so:

$$b_{t_0,\omega}(t) = b(t-t_0)e^{i\omega t}$$

whence the definition of STFT is

$$f_G(t_0,\omega) = \int_{-\infty}^{\infty} f(t)b(t-t_0)e^{-i\omega t}dt = \int_{-\infty}^{\infty} f(t)\overline{b_{t_0,\omega}(t)}dt \tag{7.8}$$

where the overbar denotes the complex conjugate. In the notation for the STFT f_G replaces $\hat{f}$. For those familiar with differential equations the G is a nod in the direction of Green's functions, for more see the excellent classic text Weinberger (1965). If the window function is real, then we get the expected STFT

$$f_G(t_0,\omega) = \int_{-\infty}^{\infty} f_b(t)e^{-i\omega t}dt.$$

The window function satisfies the weak condition

$$\hat{b}(0) = \int_{-\infty}^{\infty} b(t)dt \neq 0$$

simply by definition of a window. The definition of the left hand function is

$$\hat{b}(\omega) = \int_{-\infty}^{\infty} b(t)e^{-i\omega t}dt$$

the Fourier transform of $b(t)$. Of course we could have labelled $\hat{b}, b_G$. The windowing of the function $f(t)$ by $b(t)$ means that any frequency that corresponds to a wavelength longer than the window cannot be adequately captured, so its Fourier transform will not display frequencies that are very short. Windowing such as this is called a smoothing filter or *low pass filter*. In common language in terms of sound, they filter out hiss. Equation (7.8) can be written as an inner product of the form

$$f_G(t_0, \omega) = \langle f(t), b(t - t_0)e^{i\omega t} \rangle.$$

The window function multiplied by the complex sinusoid $b_{t_0,\omega} = b(t - t_0)e^{i\omega t}$ acts like a wave packet. The sinc function $\sin[\omega(t - t_0)]/[t - t_0]$ is the simplest manifestation of such a wave packet, see Fig. 7.2. The inverse formula for the STFT is straightforward:

$$f_b(t) = b(t - t_0)f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f_G(t_0, \omega)e^{i\omega t}d\omega.$$

If this STFT is written sightly differently with $t_0 = \tau$:

$$f_{G_b}(\tau, \xi) = \int_{-\infty}^{\infty} f(t)\overline{b_{\tau,\xi}(t)}dt = \int_{-\infty}^{\infty} f(t)b(t - \tau)e^{-i\xi t}dt = \int_{-\infty}^{\infty} f_b(t, \tau)e^{-i\xi t}dt$$

with

$$f_b(\tau, t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f_{G_b}(\tau, \xi)e^{i\xi t}d\xi$$

and the original signal $f(t)$ is recovered via the double integral:

$$f(t) = \frac{1}{2\pi||b(t)||}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{G_b}(\tau, \xi)\overline{b_{\tau,\xi}(t)}d\xi d\tau$$

then we have a general windowed Fourier transform. This is only one step away from being able to define the wavelet transform, based on a wavelet rather than a windowed function. See Goswami and Chan (1999) for more details. We shall finish this chapter with a specific example

Example 7.8 *Let $f(t) = \sin(\pi t)$ and the window function $b(t)$ be the simple symmetrical top hat function:*

$$b(t) = \begin{cases} 1, & -1 \le t \le 1 \\ 0, & \textit{otherwise} \end{cases}.$$

Determine the STFT f_G

Solution Using the given window function $b(t)$ we see immediately that

$$f_b(t) = \begin{cases} \sin(\pi t), & -1 \le t \le 1 \\ 0, & \text{otherwise} \end{cases}.$$

whence

$$f_G = \int_{-1}^{1} \sin(\pi t)e^{-i\omega t}dt = \int_{-1}^{1} \{\sin \pi t \cos \omega t - i \sin \pi t \sin \omega t\}dt.$$

Some of the tedious integration is avoided by noticing that the first (real) term on the right is an asymmetric function and so integrates to zero. The second term gives the answer

$$f_G(\omega) = i\left[\frac{\sin(\pi+\omega)}{\pi+\omega} - \frac{\sin(\pi-\omega)}{\pi-\omega}\right].$$

The original function $f(t)$ can be retrieved through the inversion formula

$$f(t) = \frac{1}{2\pi}\int_{-1}^{1}\left[\frac{\sin(\pi+\omega)}{\pi+\omega} - \frac{\sin(\pi-\omega)}{\pi-\omega}\right]\sin(\omega t)d\omega$$

but the integration is challenging.

7.9 Exercises

1. Using Father, Mother, Daughter1 and Daughter2 wavelets as a basis for all four vectors, write down the form of the four vector $[4, 8, 10, 14]^T$. Express the general vector $[a, b, c, d]^T$ for arbitrary a, b, c and d in terms of this basis. (T denotes transpose.)
2. Write down the 16 basis functions developed through the Haar wavelet and generations of daughters. Show that they are orthogonal and normalise them. Express the vector $[1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]^T$ in terms of this basis.
3. Find the Fourier transform of the kth generation daughter wavelet $\psi_{j,k}(t) = 2^{j/2}\psi\left(2^j t - k\right)$ where j, k are integers and

$$\psi(t) = \begin{cases} 1, & 0 \le t < \frac{1}{2} \\ -1, & \frac{1}{2} \le t < 1 \\ 0, & \text{otherwise.} \end{cases}$$

4. Show that

$$\int_{-\infty}^{\infty}|\psi(t)|^2 dt = \int_{-\infty}^{\infty}|\psi_{j,k}(t)|^2 dt.$$

   That is the norm of the mother wavelet is the same as that of all the daughter wavelets. What is this value?
5. Show by direct computation that the function $f(t) = e^{-\frac{1}{2}t^2}$ and its Fourier transform $\hat{f}(\omega) = \sqrt{2\pi}e^{-\frac{1}{2}\omega^2}$ give rise to equality $\Delta_f \Delta_{\hat{f}} = \frac{1}{2}$ in the Heisenberg inequality.
6. Repeat the calculation of the previous question using the function $g_\alpha(t) = \frac{1}{2\sqrt{\pi\alpha}}e^{-\frac{1}{4\alpha}t^2}$ and its Fourier transform $\hat{g}_\alpha(\omega) = e^{-\alpha\omega^2}$. Show that equality of the Heisenberg inequality is preserved for all values of α. [The function

$g_\alpha(t) = \frac{1}{2\sqrt{\pi\alpha}} e^{-\frac{1}{4\alpha}t^2}$ is used in the Gabor transform, a special case of the STFT (Short Time Fourier transform)].

7. Show that the STFT is linear. Establish also the following two results:
(a) Writing $f_1(t) = f(t - t_1)$ show that the STFT

$$f_{1G_b} = e^{-i\omega t_1} f_{G_b}(t_0 - t_1, \omega)$$

(b) Writing $f_2(t) = f(t)e^{i\omega_2 t}$ show that the STFT

$$f_{2G_b} = f_{G_b}(b, \omega - \omega_2).$$

These represent time shift and frequency shift respectively.

8. Let $f(t) = \sin(\pi t)$ and the window function $b(t)$ be the function:

$$b(t) = \begin{cases} 1 + t, & -1 \leq t < 0 \\ 1 - t, & 0 \leq t < 1 \\ 0, & \text{otherwise.} \end{cases}$$

Determine the STFT f_G.

Chapter 8
Complex Variables and Laplace Transforms

8.1 Introduction

The material in this chapter is written on the assumption that you have some familiarity with complex variable theory (or complex analysis). That is we assume that defining $f(z)$ where $z = x + iy$, $\quad i = \sqrt{-1}$, and where x and y are independent variables is not totally mysterious. In Laplace transforms, s can fruitfully be thought of as a complex variable. Indeed parts of this book (Sect. 6.2 for example) have already strayed into this territory.

For those for whom complex analysis is entirely new (rather unlikely if you have got this far), there are many excellent books on the subject. Those by Priestley (1985), Stewart and Tall (1983) or (more to my personal taste) Needham (1997) or Osborne (1999) are recommended. We give a brief resumé of required results without detailed proof. The principal reason for needing complex variable theory is to be able to use and understand the proof of the formula for inverting the Laplace transform. Section 7.6 goes a little further than this, but not much. The complex analysis given here is therefore by no means complete.

8.2 Rudiments of Complex Analysis

In this section we shall use $z(= x + iy)$ as our complex variable. It will be assumed that complex numbers, e.g. $2 + 3i$ are familiar, as are their representation on an Argand diagram (Fig. 8.1). The quantity $z = x + iy$ is different in that x and y are both (real) variables, so z must be a complex variable. In this chapter we shall be concerned with $f(z)$, that is the properties of functions of a complex variable.

In general we are able to split $f(z)$ into real and imaginary parts

$$f(z) = \phi(x, y) + i\psi(x, y)$$

P. Dyke, *An Introduction to Laplace Transforms and Fourier Series*,
Springer Undergraduate Mathematics Series, DOI: 10.1007/978-1-4471-6395-4_8,

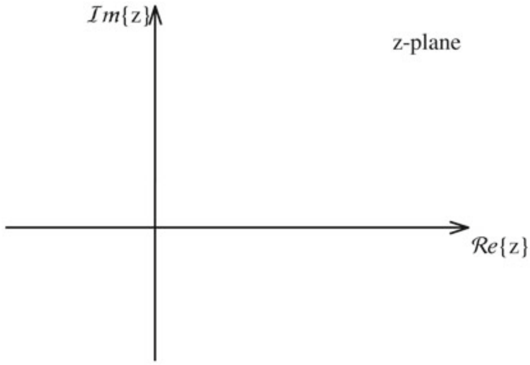


Fig. 8.1 The Argand diagram

where ϕ and ψ are real functions of the two real variables x and y. For example

$$z^2 = x^2 - y^2 + i(2xy)$$

$$\sin(z) = \sin(x)\cosh(y) + i\cos(x)\sinh(y).$$

This is the reason that many results from two real variable calculus are useful in complex analysis. However it does not by any means tell the whole story of the power of complex analysis.

One astounding fact is that if $f(z)$ is a function that is once differentiable with respect to z, then it is also differentiable twice, three times, as many times as we like. The function $f(z)$ is then called a *regular* or *analytic* function. Such a concept is absent in real analysis where differentiability to any specific order does not guarantee further differentiability. If $f(z)$ is regular then we can show that

$$\frac{\partial\phi}{\partial x} = -\frac{\partial\psi}{\partial y} \text{ and } \frac{\partial\phi}{\partial y} = \frac{\partial\psi}{\partial x}.$$

These are called the Cauchy–Riemann equations. Further, we can show that

$$\nabla^2\phi = 0, \text{ and } \nabla^2\psi = 0$$

i.e. both ϕ and ψ are harmonic. We shall not prove any of these results.

Once a function of a single real variable is deemed many times differentiable in a certain range (one dimensional domain), then one possibility is to express the function as a power series. Power series play a central role in both real and complex analysis. The power series of a function of a real variable about the point $x = x_0$ is the Taylor series. Truncated after $n + 1$ terms, this takes the form

$$f(x) = f(x_0) + (x - x_0)f'(x_0) + \frac{(x - x_0)^2}{2!}f''(x_0) + \cdots + \frac{(x - x_0)^n}{n!}f^{(n)}(x_0) + R_n$$

where

$$R_n = \frac{(x - x_0)^{n+1}}{(n+1)!} f^{(n+1)}(x_0 + \theta(x - x_0)) \quad (0 \le \theta \le 1)$$

is the remainder. This series is valid in the range $|x - x_0| \le r$, r is called the radius of convergence. The function $f(x)$ is differentiable $n + 1$ times in the region $x_0 - r \le x \le x_0 + r$.

This result carries through unchanged to complex variables

$$f(z) = f(z_0) + (z - z_0) f'(z_0) + \frac{(z - z_0)^2}{2!} f''(z_0) + \cdots + \frac{(z - z_0)^n}{n!} f^{(n)}(z_0) + \cdots$$

but now of course $f(z)$ is analytic in a *disc* $|z - z_0| < r$. What follows, however has no direct analogy in real variables. The next step is to consider functions that are regular in an *annular* region $r_1 < |z - z_0| < r_2$, or in the limit, a *punctured disc* $0 < |z - z_0| < r_2$. In such a region, a complex function $f(z)$ possesses a Laurent series

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n.$$

The $a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots$ part of the series is the "Taylor part" but the a_ns cannot in general be expressed in terms of derivatives of $f(z)$ evaluated at $z = z_0$ for the very good reason that $f(z)$ is not analytic at $z = z_0$ so such derivatives are not defined. The rest of the Laurent series

$$\cdots + \frac{a_{-2}}{(z - z_0)^2} + \frac{a_{-1}}{(z - z_0)}$$

is called the principal part of the series and is usefully employed in helping to characterise with some precision what happens to $f(z)$ at $z = z_0$. If $a_{-1}, a_{-2}, \ldots, a_{-n}, \ldots$ are all zero, then $f(z)$ *is* analytic at $z = z_0$ and possesses a Taylor series. In this case, $a_1 = f'(z_0)$, $a_2 = \frac{1}{2!} f''(z_0)$ etc. This however is not always true. The last coefficient of the principal part, a_{-1} the coefficient of $\dfrac{1}{z - z_0}$, is termed the *residue* of $f(z)$ at $z = z_0$ and has particular importance. Dealing with finding the a_ns (positive or negative n) in the Laurent series for $f(z)$ takes us into the realm of complex integration. This is because, in order to find a_n we manipulate the values $f(z)$ has in the region of validity of the Laurent series, i.e. in the annular region, to infer something about a_n. To do this, a complex version of the mean value theorem for integrals is used called Cauchy's integral formulae. In general, these take the form

$$g^{(n)}(z_0) = \frac{n!}{2\pi i} \int_C \frac{g(z)}{(z - z_0)^{n+1}} dz$$

where we assume that $g(z)$ is an analytic function of z everywhere inside C (even at $z = z_0$). (Integration in this form is formally defined in the next section.) The case $n = 1$ gives insight, as the integral then assumes classic mean value form. Now if we consider $f(z)$, which is not analytic at $z = z_0$, and let

$$f(z) = \frac{g(z)}{(z - z_0)^n}$$

where $g(z)$ *is* analytic at $z = z_0$, then we have said something about the behaviour of $f(z)$ at $z = z_0$. $f(z)$ has what is called a *singularity* at $z = z_0$, and we have specified this as a pole of order n. A consequence of $g(z)$ being analytic and hence possessing a Taylor series about $z = z_0$ valid in $|z - z_0| = r$ is that the principal part of $f(z)$ has leading term $\dfrac{a_{-n}}{(z - z_0)^n}$ with a_{-n-1}, a_{-n-2} etc. all equalling zero. A pole of order n is therefore characterised by $f(z)$ having n terms in its principal part. If $n = 1$, $f(z)$ has a simple pole (pole of order 1) at $z = z_0$. If there are infinitely many terms in the principal part of $f(z)$, then $f(z)$ is called an *essential* singularity of $f(z)$, and there are no Cauchy integral formulae. The foregoing theory is not valid for such functions. It is also worth mentioning branch points at this stage, although they do not feature for a while yet. A branch point of a function is a point at which the function is many valued. $\sqrt{z}$ is a simple example, $\mathrm{Ln}(z)$ a more complicated one. The ensuing theory is not valid for branch points.

8.3 Complex Integration

The integration of complex functions produces many surprising results, none of which are even hinted at by the integration of a single real variable. Integration in the complex z plane is a line integral. Most of the integrals we shall be concerned with are integrals around closed contours.

Suppose $p(t)$ is a complex valued function of the real variable t, with $t \in [a, b]$ which has real part $p_r(t)$ and imaginary part $p_i(t)$ both of which are piecewise continuous on $[a, b]$. We can then integrate $p(t)$ by writing

$$\int_a^b p(t)dt = \int_a^b p_r(t)dt + i\int_a^b p_i(t)dt.$$

The integrals on the right are Riemann integrals. We can now move on to define contour integration. It is possible to do this in terms of line integrals. However, line integrals have not made an appearance in this book so we avoid them here. Instead we introduce the idea of contours via the following set of simple definitions:

Definition 8.1 *1. An* arc *C is a set of points $\{(x(t), y(t)) : t \in [a, b]\}$ where $x(t)$ and $y(t)$ are continuous functions of the real variable t. The arc is conveniently described in terms of the complex valued function z of the real variable t where*

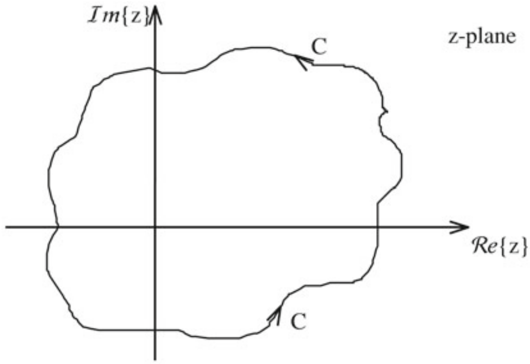


Fig. 8.2 The curve C in the complex plane

$$z(t) = x(t) + iy(t).$$

2. *An arc is* simple *if it does not cross itself. That is* $z(t_1) = z(t_2) \quad \Rightarrow \quad t_1 = t_2$ *for all* $t_1, t_2 \in [a, b]$.
3. *An arc is* smooth *if* $z'(t)$ *exists and is non-zero for* $t \in [a, b]$. *This ensures that* C *has a continuously turning tangent.*
4. *A (simple)* contour *is an arc consisting of a finite number of (simple) smooth arcs joined end to end. When only the initial and final values of* $z(t)$ *coincide, the contour is a (simple)* closed contour.

Here, all contours will be assumed to be simple. It can be seen that as t varies, $z(t)$ describes a curve (contour) on the z-plane. t is a real parameter, the value of which uniquely defines a point on the curve $z(t)$ on the z-plane. The values $t = a$ and $t = b$ give the end points. If they are the same as in Fig. 8.2 the contour is closed, and it is closed contours that are of interest here. By convention, t increasing means that the contour is described anti-clockwise. This comes from the parameter θ describing the circle $z(\theta) = e^{i\theta} = \cos(\theta) + i\sin(\theta)$ anti-clockwise as θ increases. Here is the definition of integration which follows directly from this parametric description of a curve.

Definition 8.2 *Let* C *be a simple contour as defined above extending from the point* $\alpha = z(a)$ *to* $\beta = z(b)$. *Let a domain* $\mathcal{D}$ *be a subset of the complex plane and let the curve* C *lie wholly within it. Define* $f(z)$ *to be a piecewise continuous function on* C, *that is* $f(z(t))$ *is piecewise continuous on the interval* $[a, b]$. *Then the* contour integral *of* $f(z)$ *along the contour* C *is*

$$\int_C f(z)dz = \int_a^b f(z)dz = \int_a^b f(z(t))z'(t)dt.$$

In addition to $f(z)$ being continuous at all points of a curve C (see Fig. 8.2) it will be also be assumed to be of finite length (rectifiable).

An alternative definition following classic Riemann integration (see Chap. 1) is also possible.

The first result we need is Cauchy's theorem which we now state.

Theorem 8.1 *If $f(z)$ is analytic in a domain D and on its (closed) boundary C then*

$$\oint_C f(z)dz = 0$$

where the small circle within the integral sign denotes that the integral is around a closed loop.

Proof There are several proofs of Cauchy's theorem that place no reliance on the analyticity of $f(z)$ *on* C and only use analyticity of $f(z)$ *inside* C. However these proofs are rather involved and unenlightening except for those whose principal interest is pure mathematics. The most straightforward proof makes use of Green's theorem in the plane which states that if $P(x, y)$ and $Q(x, y)$ are continuous and have continuous derivatives in D and on C, then

$$\oint_C Pdx + Qdy = \int\int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy.$$

This is easily proved by direct integration of the right hand side. To apply this to complex variables, consider $f(z) = P(x, y) + iQ(x, y)$ and $z = x + iy$ so

$$\begin{aligned} f(z)dz &= (P(x, y) + iQ(x, y))(dx + idy) \\ &= P(x, y)dx - Q(x, y)dy + i(P(x, y)dy + Q(x, y)dx) \end{aligned}$$

and so

$$\oint_C f(z)dz = \oint_C (P(x, y)dx - Q(x, y)dy) + i\oint_C (P(x, y)dy + Q(x, y)dx).$$

Using Green's theorem in the plane for both integrals gives

$$\oint_C f(z)dz = \int\int_D \left(-\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy + i\int\int_D \left(\frac{\partial P}{\partial x} - \frac{\partial Q}{\partial y}\right) dxdy.$$

Now the Cauchy–Riemann equations imply that if

$$f(z) = P(x, y) + iQ(x, y) \text{ then } \frac{\partial P}{\partial x} = \frac{\partial Q}{\partial y}, \text{ and } \frac{\partial P}{\partial y} = -\frac{\partial Q}{\partial x}$$

which immediately gives

$$\oint_C f(z)dz = 0$$

as required. □

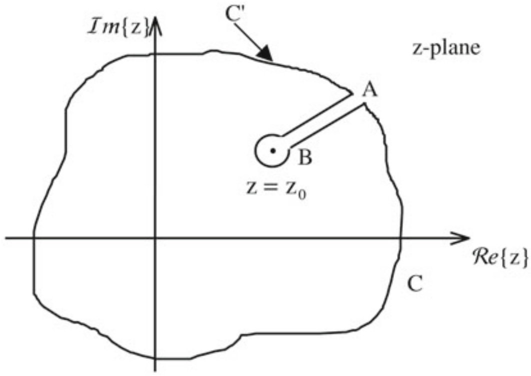


Fig. 8.3 The indented contour

Cauchy's theorem is so useful that pure mathematicians have spent much time and effort reducing the conditions for its validity. It is valid if $f(z)$ is analytic inside and not necessarily on C (as has already been said). This means that it is possible to extend it to regions that are semi-infinite (for example, a half plane). This is crucial for a text on Laplace transforms. Let us look at another consequence of Cauchy's theorem.

Suppose $f(z)$ is analytic inside a closed curve C except at the point $z = z_0$ where it is singular. Then

$$\oint_C f(z)dz$$

may or may not be zero. If we *exclude* the point $z = z_0$ by drawing the contour C' (see Fig. 8.3) where

$$\oint_{C'} = \int_C + \int_{AB} + \int_{BA} - \int_\gamma$$

and γ is a small circle surrounding $z = z_0$, then

$$\oint_{C'} f(z)dz = 0$$

as $f(z)$ *is* analytic inside C' by construction. Hence

$$\int_C f(z)dz = \int_\gamma f(z)dz$$

since

$\int_{AB} f(z)dz + \int_{BA} f(z)dz = 0$ as they are equal in magnitude but opposite in sign.

In order to evaluate $\int_C f(z)dz$ therefore we need only to evaluate $\int_\gamma f(z)dz$ where γ is a small circle surrounding the singularity $(z = z_0)$ of $f(z)$. Now we apply the theory of Laurent series introduced in the last section. In order to do this with success, we restrict attention to the case where $z = z_0$ is a pole. Inside γ, $f(z)$ can be represented by the Laurent series

$$\frac{a_{-n}}{(z-z_0)^n} + \cdots + \frac{a_{-2}}{(z-z_0)^2} + \frac{a_{-1}}{(z-z_0)} + a_0 + \sum_{k=1}^{\infty} a_k (z-z_0)^k.$$

We can thus evaluate $\int_\gamma f(z)dz$ directly term by term. This is valid as the Laurent series is uniformly convergent. A typical integral is thus

$$\int_\gamma (z-z_0)^k dz \quad k \geq -n, \quad n \text{ any positive integer or zero.}$$

On γ, $\quad z - z_0 = \epsilon e^{i\theta}, \quad 0 \leq \theta < 2\pi$ where ϵ is the radius of the circle γ and $dz = i\epsilon e^{i\theta} d\theta$. Hence

$$\begin{aligned}\int_\gamma (z-z_0)^k dz &= \int_0^{2\pi} i\epsilon^{k+1} e^{(k+1)i\theta} d\theta \\ &= \left[\frac{i\epsilon^{k+1}}{i(k+1)} e^{i(k+1)\theta}\right]_0^{2\pi} \quad \text{if } k \neq -1 \\ &= 0.\end{aligned}$$

If $k = -1$

$$\int_\gamma (z-z_0)^{-1} dz = \int_0^{2\pi} \frac{i\epsilon e^{i\theta}}{\epsilon e^{i\theta}} d\theta = 2\pi i.$$

Thus

$$\int_\gamma (z-z_0)^k dz = \begin{cases} 0 & k \neq -1 \\ 2\pi i & k = -1. \end{cases}$$

Hence

$$\int_\gamma f(z)dz = 2\pi i a_{-1}$$

where a_{-1} is the coefficient of $\frac{1}{z-z_0}$ in the Laurent series of $f(z)$ about the singularity at $z = z_0$ called the *residue* of $f(z)$ at $z = z_0$. We thus, courtesy of Cauchy's theorem, arrive at the residue theorem.

Theorem 8.2 (Residue Theorem) *If $f(z)$ is analytic within and on C except at points $z_1, z_2, \ldots, z_N$ that lie inside C where $f(z)$ has poles then*

$$\oint_C f(z)dz = 2\pi i \sum_{k=1}^{N} (sum\ of\ residues\ of\ f(z)\ at\ z = z_k).$$

Proof This result follows immediately on a straightforward generalisation of the case just considered to the case of $f(z)$ possessing N poles inside C. No further elaboration is either necessary or given. □

If $f(z)$ is a function whose only singularities are poles, it is called meromorphic. Meromorphic functions when integrated around closed contours C that contain at least one of the singularities of $f(z)$ lead to simple integrals. Distorting these closed contours into half or quarter planes leads to one of the most widespread of the applications of the residue theorem, namely the evaluation of real but improper integrals such as

$$\int_0^\infty \frac{dx}{1+x^4} \quad \text{or} \quad \int_0^\infty \frac{\cos(\pi x)}{a^2+x^2}dx.$$

These are easily evaluated by residue calculus (i.e. application of the residue theorem). The first by $\oint_C \frac{dz}{1+z^4}$ where C is a semi-circle in the upper half plane whose radius then tends to infinity and the second by considering $\oint_C \frac{e^{i\pi x}}{a^2+z^2}dz$ over a similar contour. We shall do these examples as illustrations, but point to books on complex variables for further examples. It is the evaluation of contour integrals that is a skill required for the interpretation of the inverse Laplace transform.

Example 8.1 *Use suitable contours C to evaluate the two real integrals*

(i) $\displaystyle\int_0^\infty \frac{\cos(\pi x)}{a^2+x^2}dx$

(ii) $\displaystyle\int_0^\infty \frac{dx}{1+x^4}.$

Solution

(i) For the first part, we choose the contour C shown in Fig. 8.4, that is a semi-circular contour on the upper half plane. We consider the integral

$$\oint_C \frac{e^{i\pi x}}{a^2+z^2}dz.$$

Now,

$$\oint_C \frac{e^{i\pi z}}{a^2+z^2}dz = \int_\Gamma \frac{e^{i\pi z}}{a^2+z^2}dz + \int_{-R}^{R} \frac{e^{i\pi z}}{a^2+z^2}dz$$

where Γ denotes the curved portion of C. On Γ,

$$\left|\frac{e^{i\pi z}}{z^2+a^2}\right| \leq \frac{1}{R^2+a^2}$$

hence

$$\begin{aligned}\left|\int_\Gamma \frac{e^{i\pi x}}{a^2+z^2}dz\right| &\leq \frac{1}{a^2+R^2} \times \text{length of } \Gamma \\ &= \frac{\pi R}{a^2+R^2} \to 0 \text{ as } R \to \infty.\end{aligned}$$

Hence, as $R \to \infty$

$$\oint_C \frac{e^{i\pi z}}{a^2+z^2}dz \to \int_{-\infty}^{\infty} \frac{e^{i\pi x}}{a^2+x^2}dx \text{ as on the real axis } z = x.$$

Using the residue theorem

$$\oint_C \frac{e^{i\pi z}}{a^2+z^2}dz = 2\pi i\{\text{residue at } z = ia\}.$$

The residue at $z = ia$ is given by the simple formula

$$\lim_{z\to z_0}[(z-z_0)f(z)] = a_{-1}$$

where the evaluation of the limit usually involves L'Hôpital's rule. Thus we have

$$\lim_{z\to ia}\frac{(z-ia)e^{\pi iz}}{z^2+a^2} = \frac{e^{-\pi a}}{2ai}.$$

Thus, letting $R \to \infty$ gives

$$\int_{-\infty}^{\infty}\frac{e^{i\pi x}}{a^2+x^2}dx = \frac{\pi}{a}e^{-\pi a}.$$

Finally, note that

$$\int_{-\infty}^{\infty}\frac{e^{i\pi x}}{a^2+x^2}dx = \int_{-\infty}^{\infty}\frac{\cos(\pi x)}{a^2+x^2}dx + i\int_{-\infty}^{\infty}\frac{\sin(\pi x)}{a^2+x^2}dx$$

and

$$\int_{-\infty}^{\infty}\frac{\cos(\pi x)}{a^2+x^2}dx = 2\int_0^{\infty}\frac{\cos(\pi x)}{a^2+x^2}dx.$$

Thus,

$$\int_0^\infty \frac{\cos(\pi x)}{a^2 + x^2} dx = \frac{2\pi}{a} e^{-\pi a}.$$

(ii) We could use a quarter circle ($x > 0$ and $y > 0$) for this problem as this only requires the calculation of a single residue $z = \frac{1+i}{\sqrt{2}}$. However, using the same contour (Fig. 8.4) is in the end easier. The integral

$$\int_C \frac{dz}{1 + z^4} = 2\pi i \{\text{sum of residues}\}.$$

The residues are at the two solutions of $z^4 + 1 = 0$ that lie inside C, i.e. $z = \frac{1+i}{\sqrt{2}}$ and $z = \frac{-1+i}{\sqrt{2}}$. With $f(z) = \frac{1}{z^4}$, the two values of the residues calculated using the same formula as before are $\frac{-1-i}{4\sqrt{2}}$ and $\frac{1-i}{4\sqrt{2}}$. Their sum is $\frac{-i}{2\sqrt{2}}$. Hence

$$\int_C \frac{dz}{1 + z^4} = \frac{\pi}{\sqrt{2}}$$

on Γ (the curved part of C)

$$\left| \frac{1}{1 + z^4} \right| < \frac{1}{R^4 - 1}.$$

Hence

$$\left| \int_\Gamma \frac{dz}{1 + z^4} \right| < \frac{\pi R}{R^4 - 1} \to 0 \quad \text{as} \quad R \to \infty.$$

Thus

$$\int_{-\infty}^\infty \frac{dx}{1 + x^4} = \frac{\pi}{\sqrt{2}}.$$

Hence

$$\int_0^\infty \frac{dx}{1 + x^4} = \frac{1}{2} \int_{-\infty}^\infty \frac{dx}{1 + x^4} = \frac{\pi}{2\sqrt{2}}.$$

This is all we shall do on this large topic here. We need to move on and consider branch points.

8.4 Branch Points

For the applications of complex variables required in this book, we need one further important development. Functions such as square roots are double valued for real variables. In complex variables, square roots and the like are called functions with branch points.

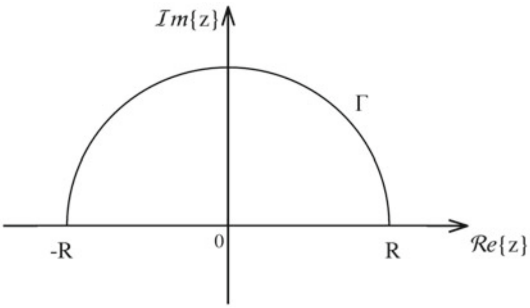


Fig. 8.4 The semi-circular contour C

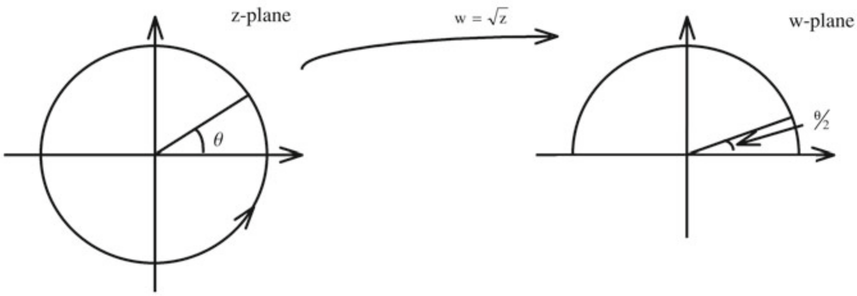


Fig. 8.5 The mapping $w = \sqrt{z}$

The function $w = \sqrt{z}$ has a branch point at $z = 0$. To see this and to get a feel for the implications, write $z = re^{i\theta}$ and as θ varies from 0 to 2π, z describes a circle radius r in the z plane, only a semi circle radius $\sqrt{r}$ is described in the w plane. In order for w to return to its original value, θ has to reach 4π. Therefore there are actually two planes superimposed upon one another in the z plane (θ varying from 0 to 4π), under the mapping each point on the w plane can arise from one point on (one of) the z planes. The two z planes are sometimes called Riemann sheets, and the place where one sheet ends and the next one starts is marked by a cut from $z = 0$ to $z = \infty$ and is typically the positive real axis (see Fig. 8.5). Perhaps a better visualisation is given in Fig. 8.6 in which the vertical axis is the imaginary part of $\sqrt{z}$. It is clearly seen that a discontinuity develops along the negative real axis, hence the cut. The function $w = \sqrt{z}$ has two sheets, whereas the function $w = z^{\frac{1}{N}}$ (N a positive integer) has N sheets and the function $w = \ln(z)$ has infinitely many sheets.

When a contour is defined for the purposes of integration, it is not permitted for the contour to cross a cut. Figure 8.7 shows a cut and a typical way in which crossing it is avoided. In this contour, a complete circuit of the origin is desired but rendered impossible because of the cut along the positive real axis. So we start at B just above the real axis, go around the circle as far as C below B. Then along CD just below

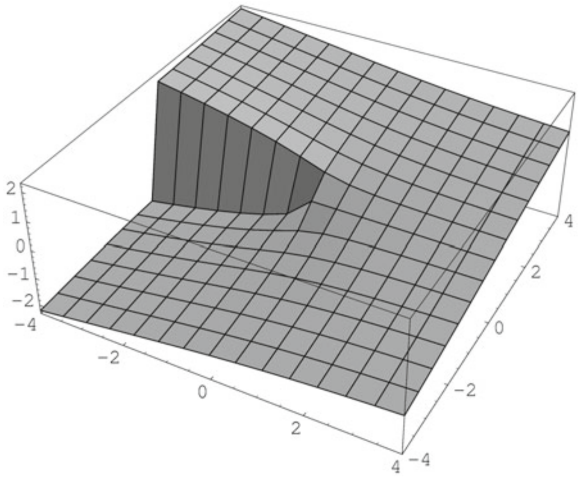
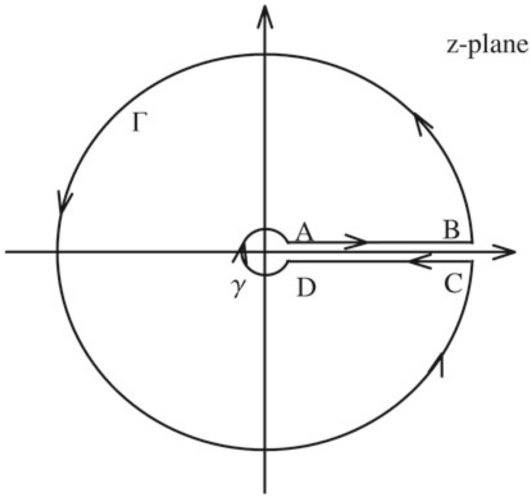


Fig. 8.6 A 3D visualisation of $w = \sqrt{z}$. The horizontal axes are the Argand diagram and the quantity $\Im\sqrt{z}$ is displayed on the vertical axis. The cut along the negative real axis is clearly visible

Fig. 8.7 The keyhole contour



the real axis, around a small circle surrounding the origin as far as A then along and just above the real axis to complete the circuit at B. In order for this contour (called a *key hole* contour) to approach the desired circular contour, $|AD| \to 0$, $|BC| \to 0$ and the radius of the small circle surrounding the origin also tends to zero. In order to see this process in operation let us do an example.

Example 8.2 *Find the value of the real integral*

$$\int_0^\infty \frac{x^{\alpha-1}}{1+x}dx \quad \textit{where} \quad 0<\alpha<1 \quad \alpha \textit{ a real constant.}$$

Solution In order to find this integral, we evaluate the complex contour integral

$$\int_C \frac{z^{\alpha-1}}{1+z}dz$$

where C is the keyhole contour shown in Fig. 8.7.

Sometimes there is trouble because there is a singularity where the cut is usually drawn. We shall meet this in the next example, but the positive real axis is free of singularities in this example. The simple pole at $z=-1$ is inside C and the residue is given by

$$\lim_{z\to -1}(z+1)\frac{z^{\alpha-1}}{(z+1)} = e^{(\alpha-1)i\pi}.$$

Thus

$$\begin{aligned}\int_C \frac{z^{\alpha-1}}{1+z}dz &= \int_\Gamma \frac{z^{\alpha-1}}{1+z}dz + \int_{CD} \frac{z^{\alpha-1}}{1+z}dz + \int_\gamma \frac{z^{\alpha-1}}{1+z}dz + \int_{AB} \frac{z^{\alpha-1}}{1+z}dz \\ &= 2\pi i e^{(\alpha-1)i\pi}.\end{aligned}$$

Each integral is taken in turn.

On Γ

$$z = Re^{i\theta}, \quad 0 \le \theta \le 2\pi, \quad R >> 1,$$

$$\left|\frac{z^{\alpha-1}}{1+z}\right| < \frac{R^{\alpha-1}}{R-1};$$

hence

$$\left|\int_C \frac{z^{\alpha-1}}{1+z}dz\right| < \frac{R^{\alpha-1}}{R-1}2\pi R \to 0 \text{ as } R\to\infty \text{ since } \alpha<1.$$

On CD, $\quad z = xe^{2\pi i}$ so

$$\int_{CD} \frac{z^{\alpha-1}}{1+z}dz = \int_R^\epsilon \frac{x^{\alpha-1}}{1+x}e^{2\pi i(\alpha-1)}dx.$$

On γ

$$z = \epsilon e^{i\theta}, \quad 0 \le \theta \le 2\pi, \quad \epsilon << 1,$$

so
$$\int_\gamma \frac{z^{\alpha-1}}{1+z}dz = \int_{2\pi}^{0} \frac{\epsilon^{\alpha-1}e^{i(\alpha-1)\theta}}{1+\epsilon e^{i\theta}} i\epsilon e^{i\theta}d\theta \to 0 \text{ as } \epsilon \to 0 \text{ since } \alpha > 0.$$

Finally, on AB, $z = x$ so

$$\int_{AB} \frac{z^{\alpha-1}}{1+z}dz = \int_\epsilon^R \frac{x^{\alpha-1}}{1+x}dx.$$

Adding all four integrals gives, letting $\epsilon \to 0$ and $R \to \infty$,

$$\int_\infty^0 \frac{x^{\alpha-1}}{1+x}e^{2\pi i(\alpha-1)}dx + \int_0^\infty \frac{x^{\alpha-1}}{1+x}dx = 2\pi i e^{(\alpha-1)i\pi}.$$

Rearranging gives
$$\int_0^\infty \frac{x^{\alpha-1}}{1+x}dx = \frac{\pi}{\sin(\alpha\pi)}.$$

This is only valid if $0 < \alpha < 1$, the integral is singular otherwise.

To gain experience with a different type of contour, here is a second example. After this, we shall be ready to evaluate the so called Bromwich contour for the inverse Laplace transform.

Example 8.3 *Use a semi-circular contour in the upper half plane to evaluate*

$$\int_C \frac{\ln z}{z^2+a^2}dz \quad (a > 0) \text{ real and positive}$$

and deduce the values of two real integrals.

Solution Figure 8.8 shows the semi-circular contour. It is indented at the origin as $z = 0$ is an essential singularity of the integrand $\dfrac{\ln z}{z^2+a^2}$. Thus

$$\int_C \frac{\ln z}{z^2+a^2}dz = 2\pi i\{\text{Residue at } z = ia\}$$

provided R is large enough and the radius of the small semi-circle $\gamma \to 0$.

The residue at $z = ia$ is given by

$$\frac{\ln(ia)}{2ia} = \frac{\pi}{4a} - i\frac{\ln(a)}{2a} \quad (a > 0)$$

so the right-hand side of the residue theorem becomes

$$\frac{\pi\ln(a)}{a} + i\frac{\pi^2}{2a}.$$

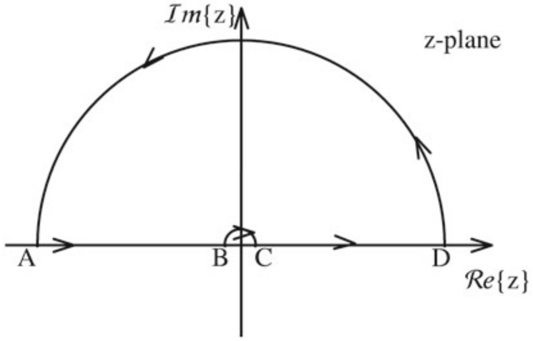


Fig. 8.8 The indented semi-circular contour C

On the semi-circular indent γ,

$$\int_\gamma \frac{\ln z}{z^2+a^2}dz = \int_\pi^0 \frac{\ln(\epsilon e^{i\theta}) i\epsilon e^{i\theta}}{\epsilon^2 e^{2i\theta}+a^2}d\theta \to 0 \text{ as } \epsilon \to 0.$$

(This is because $\epsilon \ln \epsilon \to 0$ as $\epsilon \to 0$.) Thus

$$\int_C \frac{\ln z}{z^2+a^2}dz = \frac{\pi \ln a}{a} + i\frac{\pi^2}{2a}.$$

Now, as we can see from Fig. 8.8,

$$\int_C \frac{\ln z}{z^2+a^2}dz = \int_\Gamma \frac{\ln z}{z^2+a^2}dz + \int_{AB} \frac{\ln z}{z^2+a^2}dz + \int_\gamma \frac{\ln z}{z^2+a^2}dz + \int_{CD} \frac{\ln z}{z^2+a^2}dz.$$

On Γ, $\quad z = Re^{i\theta}, \quad 0 \le \theta \le \pi$, and

$$\left|\frac{\ln z}{z^2+a^2}\right| \le \frac{\ln R}{R^2-a^2}$$

so

$$\left|\int_\Gamma \frac{\ln z}{z^2+a^2}dz\right| \le \frac{\pi R \ln R}{R^2-a^2} \to 0 \text{ as } R \to \infty.$$

Thus letting $R \to \infty$, the radius of $\gamma \to 0$ and evaluating the straight line integrals via $z = x$ on CD and $z = xe^{i\pi}$ on AB gives

$$\int_C \frac{\ln z}{z^2+a^2}dz = \int_\infty^0 \frac{\ln xe^{i\pi}}{x^2+a^2}e^{i\pi}dx + \int_0^\infty \frac{\ln x}{x^2+a^2}dx$$
$$= 2\int_0^\infty \frac{\ln x}{x^2+a^2}dx + i\pi\int_0^\infty \frac{dx}{x^2+a^2}.$$

The integral is equal to

$$\frac{\pi \ln a}{a} + i\frac{\pi^2}{2a}$$

so equating real and imaginary parts gives the two real integrals

$$\int_0^\infty \frac{\ln x}{x^2+a^2}dx = \frac{\pi \ln a}{2a}$$

and

$$\int_0^\infty \frac{dx}{x^2+a^2} = \frac{\pi}{2a}.$$

The second integral is an easily evaluated arctan standard form.

8.5 The Inverse Laplace Transform

We are now ready to derive and use the formula for the inverse Laplace transform. It is a surprise to engineers that the inverse of a transform so embedded in real variables as the Laplace transform requires so deep a knowledge of complex variables for its evaluation. It should not be so surprising having studied the last three chapters. We state the inverse transform as a theorem.

Theorem 8.3 *If the Laplace transform of $F(t)$ exists, that is $F(t)$ is of exponential order and*

$$f(s) = \int_0^\infty e^{-st}F(t)dt$$

then

$$F(t) = \lim_{k\to\infty}\left\{\frac{1}{2\pi i}\int_{\sigma-ik}^{\sigma+ik} f(s)e^{st}ds\right\} \quad t > 0$$

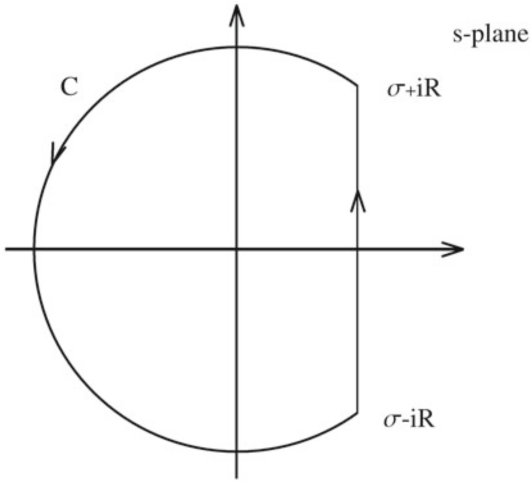
where $|F(t)| \le e^{Mt}$ for some positive real number M and σ is another real number such that $\sigma > M$.

Proof The proof of this has already been outlined in Sect. 6.2 of Chap. 6. However, we have now done enough formal complex variable theory to give a more complete proof. The outline remains the same in that we define $F_k(\omega)$ as in that chapter, namely

$$F_k(\omega) = \int_0^\infty e^{-(k+i\omega)x}f(x)dx$$

and rewrite this in notation more suited to Laplace transforms, i.e. x becomes t, $k+i\omega$ becomes s and the functions are renamed. $f(x)$ becomes $F(t)$ and $F_k(\omega)$ becomes $f(s)$. However, the mechanics of the proof follows as before with Eq. 6.2.

Fig. 8.9 The Bromwich contour



Using the new notation, these two equations convert to

$$f(s) = \int_0^\infty e^{-st} F(t) dt$$

and

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{st} f(s) d\{\Im(s)\} = \begin{cases} 0 & t < 0 \\ F(t) & t > 0 \end{cases}$$

where the integral on the left is a real integral. $d\{\Im(s)\} = d\omega$ a real differential. Converting this to a complex valued integral, formally done by recognising that $ds = id\{\Im(s)\}$, gives the required formula, viz.

$$F(t) = \frac{1}{2\pi} \int_{s_0}^{s_1} e^{st} f(s) ds$$

where s_0 and s_1 represent the infinite limits ($k - i\infty,\ k + i\infty$ in the notation of Chap. 6). Now, however we can be more precise. The required behaviour of $F(t)$ means that the real part of s must be at least as large as σ, otherwise $|F(t)|$ does not $\to 0$ as $t \to \infty$ on the straight line between s_0 and s_1. This line is parallel to the imaginary axis. The theorem is thus formally established. $\square$

The way of evaluating this integral is via a closed contour of the type shown in Fig. 8.9. This contour, often called the Bromwich contour, consists of a portion of a circle, radius R, together with a straight line segment connecting the two points $\sigma - iR$ and $\sigma + iR$. The real number σ must be selected so that all the singularities of the function $f(s)$ are to the left of this line. This follows from the conditions of Theorem 8.3. The integral

$$\int_C f(s)e^{st}ds$$

where C is the Bromwich contour is evaluated using Cauchy's residue theorem, perhaps with the addition of one or two cuts. The integral itself is the sum of the integral over the curved portion, the integral along any cuts present and

$$\int_{\sigma-iR}^{\sigma+iR} f(s)e^{st}ds$$

and the whole is $2\pi i$ times the sum of the residues of $f(s)e^{st}$ inside C. The above integral is made the subject of this formula, and as $R \to \infty$ this integral becomes $F(t)$. In this way, $F(t)$ is calculated. Let us do two examples to see how the process operates.

Example 8.4 *Use the Bromwich contour to find the value of*

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+1)(s-2)^2}\right\}.$$

Solution It is quite easy to find this particular inverse Laplace transform using partial fractions as in Chap. 2; however it serves as an illustration of the use of the contour integral method. In the next example, there are no alternative direct methods.

Now,

$$\int_C \frac{e^{st}}{(s+1)(s-2)^2}ds = 2\pi i\{\text{sum of residues}\}$$

where C is the Bromwich contour of Fig. 8.9. The residue at $s = 1$ is given by

$$\lim_{s\to -1}(s+1)\frac{e^{st}}{(s+1)(s-2)^2} = \frac{1}{9}e^{-t}.$$

The residue at $s = 2$ is given by

$$\lim_{s\to 2}\frac{d}{ds}\frac{e^{st}}{(s+1)} = \left[\frac{(s+1)te^{st} - e^{st}}{(s+1)^2}\right]_{s=2} = \frac{1}{9}(3te^{2t} - e^{2t}).$$

Thus

$$\int_C \frac{e^{st}}{(s+1)(s-2)^2}ds = 2\pi i\left\{\frac{1}{9}(e^{-t} + 3te^{2t} - e^{2t})\right\}.$$

Now,

$$\int_C \frac{e^{st}}{(s+1)(s-2)^2}ds = \int_\Gamma \frac{e^{st}}{(s+1)(s-2)^2}ds + \int_{\sigma-iR}^{\sigma+iR} \frac{e^{st}}{(s+1)(s-2)^2}ds$$

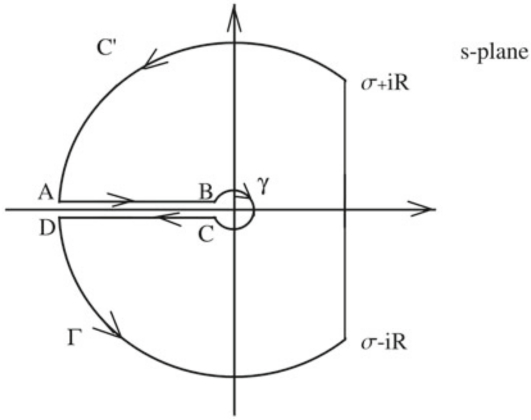


Fig. 8.10 The cut Bromwich contour

and the first integral $\to 0$ as the radius of the Bromwich contour $\to \infty$. Thus

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+1)(s-2)^2}\right\} = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{e^{st}}{(s+1)(s-2)^2}ds = \frac{1}{9}(e^{-t}+3te^{2t}-e^{2t}).$$

A result easily confirmed by the use of partial fractions.

Example 8.5 *Find*

$$\mathcal{L}^{-1}\left\{\frac{e^{-a\sqrt{s}}}{s}\right\}$$

where a is a real constant.

Solution The presence of $e^{-a\sqrt{s}}$ means that the origin $s=0$ is a branch point. It is thus necessary to use the cut Bromwich contour C' as shown in Fig. 8.10.

Now,

$$\int_{C'}\frac{e^{st-a\sqrt{s}}}{s}ds = 0$$

by Cauchy's theorem as there are no singularities of the integrand inside C'. We now evaluate the contour integral by splitting it into its five parts:-

$$\int_{C'}\frac{e^{st-a\sqrt{s}}}{s}ds = \int_{\Gamma} + \int_{\sigma-iR}^{\sigma+iR} + \int_{AB} + \int_{\gamma} + \int_{CD} = 0$$

and consider each bit in turn. The radius of the large circle Γ is R and the radius of the small circle γ is ϵ. On Γ, $s = Re^{i\theta} = R\cos(\theta) + iR\sin(\theta)$ and on the left hand side of the s-plane $\cos\theta < 0$. This means that, on Γ,

$$\left|\frac{e^{st-a\sqrt{s}}}{s}\right| = \left|\frac{e^{Rt\cos(\theta)-a\sqrt{R}\cos(\frac{1}{2}\theta)}}{R}\right|$$

and by the estimation lemma,

$$\int_{\Gamma} \to 0 \ \text{ as } \ R \to \infty.$$

The second integral is the one required and we turn our attention to the third integral along AB. On AB, $s = xe^{i\pi}$ whence

$$\frac{e^{st-a\sqrt{s}}}{s} = -\frac{e^{-xt-ai\sqrt{x}}}{x}$$

whereas on CD $s = -x$ so that

$$\frac{e^{st-a\sqrt{s}}}{s} = \frac{e^{-xt+ai\sqrt{x}}}{x}.$$

This means that

$$\int_{AB} + \int_{CD} = \int_{R}^{\epsilon} \frac{e^{-xt-ai\sqrt{x}}}{x} dx + \int_{\epsilon}^{R} \frac{e^{-xt+ai\sqrt{x}}}{x} dx.$$

It is the case that if the cut is really necessary, then integrals on either side of it never cancel. The final integral to consider is that around γ. On γ $s = \epsilon e^{i\theta}$, $-\pi \le \theta < \pi$, so that

$$\int_{\gamma} \frac{e^{-st-a\sqrt{s}}}{s} ds = \int_{\pi}^{-\pi} \frac{e^{-\epsilon e^{i\theta}t-a\sqrt{\epsilon}e^{\frac{1}{2}i\theta}}}{\epsilon e^{i\theta}} i\epsilon e^{i\theta} d\theta.$$

Now, as $\epsilon \to \ 0$

$$\int_{\gamma} \to \int_{\pi}^{-\pi} i d\theta = -2\pi i.$$

Hence, letting $R \to \infty$ and $\epsilon \to 0$ gives

$$\begin{aligned}\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \frac{e^{st-a\sqrt{s}}}{s} ds &= 1 - \frac{1}{2\pi i}\left\{\int_{\infty}^{0} \frac{e^{-xt-ai\sqrt{x}}}{x} dx + \int_{0}^{\infty} \frac{e^{-xt+ai\sqrt{x}}}{x} dx\right\} \\ &= 1 - \frac{1}{\pi}\int_{0}^{\infty} \frac{e^{-xt}\sin(a\sqrt{x})}{x} dx.\end{aligned}$$

As the left hand side is $F(t)$, the required inverse is

$$1 - \frac{1}{\pi}\int_0^\infty \frac{e^{-xt}\sin(a\sqrt{x})}{x}dx.$$

This integral can be simplified (if that is the correct word) by the substitution $x = u^2$ then using differentiation under the integral sign. Omitting the details, it is found that

$$\frac{1}{\pi}\int_0^\infty \frac{e^{-xt}\sin(a\sqrt{x})}{x}dx = \operatorname{erf}\left(\frac{a}{2\sqrt{t}}\right)$$

where erf is the error function (see Chap. 3). Hence

$$\mathcal{L}^{-1}\left\{\frac{e^{-a\sqrt{s}}}{s}\right\} = 1 - \operatorname{erf}\left(\frac{a}{2\sqrt{t}}\right) = \operatorname{erfc}\left(\frac{a}{2\sqrt{t}}\right)$$

where

$$\operatorname{erfc}(p) = \frac{2}{\sqrt{\pi}}\int_p^\infty e^{-t^2}dt$$

is the complementary error function.

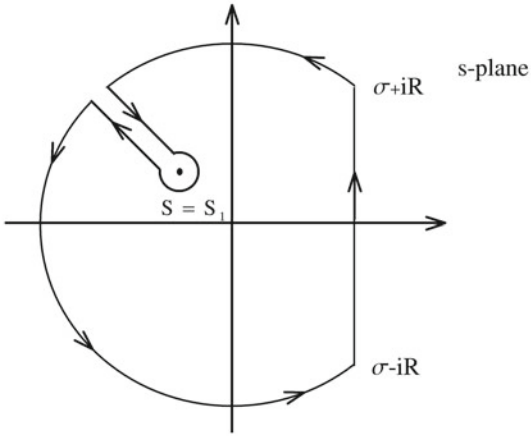
8.6 Using the Inversion Formula in Asymptotics

We saw in Chap. 5 how it is possible to use asymptotic expansions in order to gain information from a differential equation even if it was not possible to solve it in closed form. Let us return to consider this kind of problem, now armed with the complex inversion formula. It is often possible to approximate the inversion integral using asymptotic analysis when exact inversion cannot be done. Although numerical techniques are of course also applicable, asymptotic methods are often more appropriate. After all, there is little point employing numerical methods at this late stage when it would have probably been easier to use numerical techniques from the outset on the original problem. Of course, by so doing all the insight gained by adopting analytical methods would have been lost. Not wishing to lose this insight, we press on with asymptotics. The following theorem embodies a particularly useful asymptotic property for functions that have branch points.

Theorem 8.4 *If $\bar{f}(s)$ is $O(1/|s|)$ as $s \to \infty$, the singularity with largest real part is at $s = s_1$, and in some neighbourhood of s_1*

$$\bar{f}(s) = (s - s_1)^k \sum_{n=0}^\infty a_n(s - s_1)^n \quad -1 < k < 0,$$

Fig. 8.11 The contour C distorted around the branch point at $s = s_1$



then

$$f(t) = -\frac{1}{\pi}e^{s_1 t}\sin(k\pi)\sum_{n=0}^{\infty} a_n(-1)^n \frac{\Gamma(n+k+1)}{t^{n+k+1}} \text{ as } t \to \infty.$$

Proof In this proof we shall assume that $s = s_1$ is the only singularity and that it is a branch point. This is acceptable as we are seeking to prove an asymptotic result and as long as the singularities of $\bar{f}(s)$ are isolated only the one with the largest real part is of interest. If there is a tie, each singularity is considered and the contributions from each may be added. When the Bromwich contour is used, and the function $\bar{f}(s)$ has branch points, it has to be distorted around these points and a typical distortion is shown in Fig. 8.11 and the object is to approximate the integral

$$\frac{1}{2\pi i}\int_C e^{st}\bar{f}(s)ds.$$

On the lower Riemann sheet

$$s = s_1 + xe^{-i\pi}$$

and on the upper Riemann sheet

$$s = s_1 + xe^{i\pi}.$$

The method used to evaluate the integral is to invoke Watson's lemma as follows

$$f(t) = \frac{1}{2\pi i}\int_\infty^0 e^{(s_1-x)t}\bar{f}(s_1 + xe^{-i\pi})e^{-i\pi}dx + \frac{1}{2\pi i}\int_0^\infty e^{(s_1-x)t}\bar{f}(s_1 + xe^{i\pi})e^{i\pi}dx.$$

These integrals combine to give

$$f(t) = -\frac{1}{2\pi i}\int_0^\infty e^{(s_1-x)t}[\bar{f}(s_1 + xe^{i\pi}) - \bar{f}(s_1 + xe^{-i\pi})]dx.$$

Watson's lemma is now used to expand the two functions in the square bracket in powers of x. It really is just algebra, so only the essential steps are given.

Assume that $\bar{f}(s)$ can be expressed in the manner indicated in the theorem so

$$\bar{f}(s_1 + xe^{i\pi}) = (xe^{i\pi})^k \sum_{n=0}^{\infty} a_n (xe^{i\pi})^n$$

and

$$\bar{f}(s_1 + xe^{-i\pi}) = (xe^{-i\pi})^k \sum_{n=0}^{\infty} a_n (xe^{-i\pi})^n.$$

These two expressions are subtracted so that as is typical, it is the $e^{\pm ik\pi}$ terms that prevent complete cancellation. Thus

$$\bar{f}(s_1 + xe^{i\pi}) - \bar{f}(s_1 + xe^{-i\pi}) = x^k 2i\sin(k\pi) \sum_{n=0}^{\infty} a_n (-1)^n x^n.$$

We are now faced with the actual integration, so

$$\frac{1}{2\pi i}\int_0^\infty e^{(s_1-x)t}[\bar{f}(s_1 + xe^{i\pi}) - \bar{f}(s_1 + xe^{-i\pi})]dx$$

$$= \frac{\sin(k\pi)}{\pi}\int_0^\infty \sum_{n=0}^{\infty} a_n (-1)^n x^{n+k} e^{(s_1-x)t} dx$$

$$= \frac{\sin(k\pi)}{\pi} e^{s_1 t} \sum_{n=0}^{\infty} a_n (-1)^n \int_0^\infty x^{n+k} e^{-xt} dx$$

provided exchanging summation and integral signs is valid. Now the integral definition of the gamma function

$$\Gamma(n) = \int_0^\infty u^{n-1} e^{-u} du$$

comes to our aid to evaluate the integral that remains. Using the substitution $u = xt$, we have

$$\int_0^\infty x^{n+k} e^{-xt} dx = \frac{1}{t^{n+k+1}}\Gamma\{n + k + 1\}.$$

This then formally establishes the theorem, leaving only one or two niggly questions over the legality of some of the steps. One common query is how is it justifiable to integrate with respect to x as far as infinity when we are supposed to be "close to s_1" the branch point of $\bar{f}(s)$? It has to be remembered that this is an approximation valid for large t, hence the major contribution will come from that part of the contour that is near to $x = 0$. The presence of infinity in the limit of the integral is therefore somewhat misleading. (For those familiar with boundary layers in fluid mechanics, it is not uncommon for "infinity" to be as small as 0.2!) □

If the singularity at $s = s_1$ is a pole, then the integral can be evaluated directly by use of the residue theorem, giving

$$\begin{aligned} f(t) &= \frac{1}{2\pi i}\int_C e^{st}\bar{f}(s)ds \\ &= Re^{s_1 t} \end{aligned}$$

where R is the residue of $\bar{f}(s)$ at $s = s_1$. The following example illustrates the use of this theorem to approximate the behaviour of the solution to a BVP for large y.

Example 8.6 *Find the asymptotic behaviour as $y \to \infty$ for fixed x of the solution of the partial differential equation*

$$\frac{\partial^2\phi}{\partial y^2} = \frac{\partial^2\phi}{\partial x^2} - \phi, \quad x > 0, \quad y > 0,$$

such that

$$\phi(x,0) = \frac{\partial\phi}{\partial y}(x,0) = 0, \quad \phi(0,y) = 1.$$

Solution Taking the Laplace transform of the given equation with respect to y using the by now familiar notation leads to the following ODE for $\bar{\phi}$:

$$(1+s^2)\bar{\phi} = \frac{d^2\bar{\phi}}{dx^2}$$

where the boundary conditions at $y = 0$ have already been utilised. At $x = 0$ we require that $\phi = 1$ for all y. This transforms to $\bar{\phi} = 1/s$ at $x = 0$ which thus gives the solution

$$\bar{\phi}(x,s) = \frac{e^{-x\sqrt{1+s^2}}}{s}.$$

We have discarded the part of the complementary function $e^{x\sqrt{1+s^2}}$ as it does not tend to zero for large s and so cannot represent a function of s that can have arisen from a Laplace transform (see Chap. 2). The problem would be completely solved if we could invert the Laplace transform

$$\bar{u} = \frac{1}{s} e^{-x\sqrt{1+s^2}}$$

but alas this is not possible in closed form. The best we can do is to use Theorem 8.4 to find an approximation valid for large values of y. Now, $\bar{u}$ has a simple pole at $s = 0$ and two branch points at $s = \pm i$. As the real part of all of these singularities is the same, viz. zero, all three contribute to the leading term for large y. At $s = 0$ the residue is straightforwardly e^{-x}. Near $s = i$ the expansion

$$\bar{\phi}(x, s) = \frac{1}{i}[1 - (2i)^{1/2}(s - i)^{1/2}x + \cdots]$$

is valid. Near $s = -i$ the expansion

$$\bar{\phi}(x, s) = -\frac{1}{i}[1 - (-2i)^{1/2}(s + i)^{1/2}x + \cdots]$$

is valid. The value of $\phi(x, y)$, which is precisely the integral

$$\phi(x, y) = \frac{1}{2\pi i}\int_C e^{sy}\bar{\phi}(x, s)ds$$

is approximated by the following three terms; e^{-x} from the pole at $s = 0$, and the two contribution from the two branch points at $s = \pm i$. Using Theorem 8.4 these are

$$u \sim \frac{1}{\pi} e^{iy} \sin\left(\frac{1}{2}\pi\right)\left((2i)^{1/2}\frac{\Gamma\{3/2\}}{y^{3/2}}\right) \text{ near } s = i$$

and

$$u \sim \frac{1}{\pi} e^{-iy} \sin\left(-\frac{1}{2}\pi\right)\left((-2i)^{1/2}\frac{\Gamma\{3/2\}}{y^{3/2}}\right) \text{ near } s = -i.$$

The sum of all these dominant terms is

$$u \sim e^{-x} + \frac{2^{1/2}x}{\pi^{1/2}y^{3/2}}\cos\left(y + \frac{\pi}{4}\right).$$

This is the behaviour of the solution $\phi(x, y)$ for large values of y.

Further examples of the use of asymptotics can be found in specialist texts on partial differential equations, e.g. Williams (1980), Weinberger (1965). For more about asymptotic expansions, especially the rigorous side, there is nothing to better the classic text of Copson (1967).

8.7 Exercises

1. The following functions all have simple poles. Find their location and determine the residue at each pole.

$$\text{(i)}\quad \frac{1}{1+z}, \qquad \text{(ii)}\quad \frac{2z+1}{z^2-z-2}$$
$$\text{(iii)}\quad \frac{3z^2+2}{(z-1)(z^2+9)} \qquad \text{(iv)}\quad \frac{3+4z}{z^3+3z^2+2z}$$
$$\text{(v)}\quad \frac{\cos(z)}{z} \qquad \text{(vi)}\quad \frac{z}{\sin(z)}$$
$$\text{(vii)}\quad \frac{e^z}{\sin(z)}.$$

2. The following functions all have poles. Determine their location, order, and all the residues.

$$\text{(i)}\quad \left(\frac{z+1}{z-1}\right)^2 \quad \text{(ii)}\quad \frac{1}{(z^2+1)^2}$$
$$\text{(iii)}\quad \frac{\cos(z)}{z^3} \qquad \text{(iv)}\quad \frac{e^z}{\sin^2(z)}.$$

3. Use the residue theorem to evaluate the following integrals:
(i)
$$\int_C \frac{2z}{(z-1)(z+2)(z+i)}dz$$
where C is any contour that includes within it the points $z = 1,\ z = 2$ and $z = -i$.
(ii)
$$\int_C \frac{z^4}{(z-1)^3}dz$$
where C is any contour that encloses the point $z = 1$.
(iii)
$$\int_0^\infty \frac{1}{x^6+1}dx.$$
(iv)
$$\int_0^\infty \frac{\cos(2\pi x)}{x^4+x^2+1}dx.$$

4. Use the indented semi-circular contour of Fig. 8.8 to evaluate the three real integrals:

$$\text{(i)} \int_0^\infty \frac{(\ln x)^2}{x^4+1}dx, \quad \text{(ii)} \int_0^\infty \frac{\ln x}{x^4+1}dx, \quad \text{(iii)} \int_0^\infty \frac{x^\lambda}{x^2+1}dx, \quad -1 < \lambda < 1.$$

5. Determine the following inverse Laplace transforms:

$$\text{(i)}\ \mathcal{L}^{-1}\left\{\frac{1}{s\sqrt{s+1}}\right\}, \text{(ii)}\ \mathcal{L}^{-1}\left\{\frac{1}{1+\sqrt{s+1}}\right\},$$
$$\text{(iii)}\ \mathcal{L}^{-1}\left\{\frac{1}{\sqrt{s}+1}\right\}, \text{(iv)}\ \mathcal{L}^{-1}\left\{\frac{1}{\sqrt{s}-1}\right\}.$$

You may find that the integrals

$$\int_0^\infty \frac{e^{-xt}}{(x+1)\sqrt{x}}dx = -\pi e^t[-1+\operatorname{erf}\sqrt{t}]$$

and

$$\int_0^\infty \frac{e^{-xt}\sqrt{x}}{(x+1)}dx = \sqrt{\frac{\pi}{t}} - \pi e^t \operatorname{erfc}\sqrt{t}$$

help the algebra.

6. Define the function $\varphi(t)$ via the inverse Laplace transform

$$\varphi(t) = \mathcal{L}^{-1}\left\{\operatorname{erf}\left(\frac{1}{s}\right)\right\}.$$

Show that

$$\mathcal{L}\{\varphi(t)\} = \frac{2}{\sqrt{\pi s}}\sin\left(\frac{1}{\sqrt{s}}\right).$$

7. The zero order Bessel function can be defined by the series

$$J_0(xt) = \sum_{k=0}^\infty \frac{(-1)^k(\frac{1}{2}x)^{2k}t^{2k}}{(k!)^2}.$$

Show that

$$\mathcal{L}\{J_0(xt)\} = \frac{1}{s}\left(1+\frac{x^2}{s^2}\right)^{-1/2}.$$

8. Determine

$$\mathcal{L}^{-1}\left\{\frac{\cosh(x\sqrt{s})}{s\cosh(\sqrt{s})}\right\}$$

by direct use of the Bromwich contour.

9. Use the Bromwich contour to show that

$$\mathcal{L}^{-1}\{e^{-s^{1/3}}\} = \frac{3}{\pi}\int_0^\infty u^2 e^{-tu^3 - \frac{1}{2}u} \sin\left(\frac{u\sqrt{3}}{2}\right) du.$$

10. The function $\phi(x, t)$ satisfies the partial differential equation

$$\frac{\partial^2 \phi}{\partial x^2} - \frac{\partial^2 \phi}{\partial t^2} + \phi = 0$$

with

$$\phi(x, 0) = \frac{\partial \phi}{\partial t}(x, 0) = 0, \quad \phi(0, t) = t.$$

Use the asymptotic method of Sect. 8.6 to show that

$$\phi(x, t) \sim \frac{xe^t}{\sqrt{2\pi t^3}} \quad \text{as} \quad t \to \infty$$

for fixed x.

Appendix A
Answers to Exercises

Solutions

Exercises 1.4

1. (a) $\ln t$ is singular at $t = 0$, hence one might assume that the Laplace transform does not exist. However, for those familiar with the Gamma function consider the result

$$\int_0^\infty e^{-st} t^k dt = \frac{\Gamma(k+1)}{s^{k+1}}$$

which is standard and valid for non integer k. Differentiate this result with respect to k to obtain

$$\int_0^\infty e^{-st} t^k \ln t dt = \frac{\Gamma'(k+1) - \Gamma(k+1)\ln s}{s^{k+1}}.$$

Let $k = 0$ in this result to obtain

$$\int_0^\infty e^{-st} \ln t dt = \frac{\Gamma'(1) - \ln s}{s} = -\frac{\gamma + \ln s}{s}$$

where γ is the Euler-Mascheroni constant ($= 0.5772156649\ldots$). The right hand side is the Laplace transform of $\ln t$, so it does exist. The apparent singularity is in fact removable. (c.f. The Laplace transform of $t^{-\frac{1}{2}}$ also exists and is a finite quantity.)

(b)

$$\mathcal{L}\{e^{3t}\} = \int_0^\infty e^{3t} e^{-st} dt = \left[\frac{1}{3-s} e^{(3-s)t}\right]_0^\infty = \frac{1}{s-3}.$$

(c) $e^{t^2} > |e^{Mt}|$ for any M for large enough t, hence the Laplace transform does not exist (not of exponential order).

P. Dyke, *An Introduction to Laplace Transforms and Fourier Series*,
Springer Undergraduate Mathematics Series, DOI: 10.1007/978-1-4471-6395-4,

(d) the Laplace transform does not exist (singular at $t = 0$).
(e) the Laplace transform does not exist (singular at $t = 0$).
(f) does not exist (infinite number of (finite) jumps), also not defined unless t is an integer.

2. Using the definition of Laplace transform in each case, the integration is reasonably straightforward:

$$\text{(a)} \quad \int_0^\infty e^{kt}e^{-st}dt = \frac{1}{s-k}$$

as in part (b) of the previous question.
(b) Integrating by parts gives,

$$\mathcal{L}\{t^2\} = \int_0^\infty t^2e^{-st}dt = \left[-\frac{t^2}{s}e^{-st}\right]_0^\infty + \int_0^\infty \frac{2t}{s}e^{-st}dt = \frac{2}{s}\int_0^\infty te^{-st}dt.$$

Integrating by parts again gives the result $\frac{2}{s^3}$.
(c) Using the definition of $\cosh t$ gives

$$\mathcal{L}\{\cosh t\} = \frac{1}{2}\left\{\int_0^\infty e^te^{-st}dt + \int_0^\infty e^{-t}e^{-st}dt\right\}$$

$$= \frac{1}{2}\left\{\frac{1}{s-1} + \frac{1}{s+1}\right\} = \frac{s}{s^2-1}.$$

3. (a) This demands use of the first shift theorem, Theorem 1.2, which with $b = 3$ is

$$\mathcal{L}\{e^{-3t}F(t)\} = f(s+3)$$

and with $F(t) = t^2$, using part (b) of the last question gives the answer $\frac{2}{(s+3)^3}$.
(b) For this part, we use Theorem 1.1 (linearity) from which the answer

$$\frac{4}{s^2} + \frac{6}{s-4}$$

follows at once.
(c) The first shift theorem with $b = 4$ and $F(t) = \sin(5t)$ gives

$$\frac{5}{(s+4)^2+25} = \frac{5}{s^2+8s+41}.$$

4. When functions are defined in a piecewise fashion, the definition integral for the Laplace transform is used and evaluated directly. For this problem we get

$$\mathcal{L}\{F(t)\} = \int_0^\infty e^{-st} F(t)dt = \int_0^1 te^{-st}dt + \int_1^2 (2-t)e^{-st}dt$$

which after integration by parts gives

$$\frac{1}{s^2}(1-e^{-s})^2.$$

5. Using Theorem 1.3 we get

$$\text{(a)} \quad \mathcal{L}\{te^{2t}\} = -\frac{d}{ds}\frac{1}{(s-2)} = \frac{1}{(s-2)^2}$$

$$\text{(b)} \quad \mathcal{L}\{t\cos(t)\} = -\frac{d}{ds}\frac{s}{1+s^2} = \frac{-1+s^2}{(1+s^2)^2}$$

The last part demands differentiating twice,

$$\text{(c)} \quad \mathcal{L}\{t^2\cos(t)\} = \frac{d^2}{ds^2}\frac{s}{1+s^2} = \frac{2s^3-6s}{(1+s^2)^3}.$$

6. These two examples are not difficult: the first has application to oscillating systems and is evaluated directly, the second needs the first shift theorem with $b = 5$.

$$\text{(a)} \quad \mathcal{L}\{\sin(\omega t + \phi)\} = \int_0^\infty e^{-st}\sin(\omega t + \phi)dt$$

and this integral is evaluated by integrating by parts twice using the following trick. Let

$$I = \int_0^\infty e^{-st}\sin(\omega t + \phi)dt$$

then derive the formula

$$I = \left[-\frac{1}{s}e^{-st}\sin(\omega t + \phi) - \frac{\omega}{s^2}e^{-st}\cos(\omega t + \phi)\right]_0^\infty - \frac{\omega^2}{s^2}I$$

from which

$$I = \frac{s\sin(\phi) + \omega\cos(\phi)}{s^2+\omega^2}.$$

$$\text{(b)} \quad \mathcal{L}\{e^{5t}\cosh(6t)\} = \frac{s-5}{(s-5)^2-36} = \frac{s-5}{s^2-10s-11}.$$

7. This problem illustrates the difficulty in deriving a linear translation plus scaling property for Laplace transforms. The zero in the bottom limit is the culprit. Direct integration yields:

$$\mathcal{L}\{G(t)\} = \int_0^\infty e^{-st}G(t)dt = \int_{-b/a}^\infty ae^{(u-b)s/a}F(u)du$$

where we have made the substitution $t = au + b$ so that $G(t) = F(u)$. In terms of $\bar{f}(as)$ this is

$$ae^{-sb}\bar{f}(as) + ae^{-sb}\int_{-b/a}^0 e^{-ast}F(t)dt.$$

8. The proof proceeds by using the definition as follows:

$$\mathcal{L}\{F(at)\} = \int_0^\infty e^{-st}F(at)dt = \int_0^\infty e^{-su/a}F(u)du/a$$

which gives the result. Evaluation of the two Laplace transforms follows from using the results of Exercise 5 alongside the change of scale result just derived with, for (a) $a = 6$ and for (b) $a = 7$. The answers are

$$\text{(a)}\frac{-36+s^2}{(36+s^2)^2},\ \text{(b)}\frac{2s(s^2-147)}{(s^2+49)^3}.$$

Exercises 2.8

1. If $F(t) = \cos(at)$ then $F'(t) = -a\sin(at)$. The derivative formula thus gives

$$\mathcal{L}\{-a\sin(at)\} = s\mathcal{L}\{\cos(at)\} - F(0).$$

Assuming we know that $\mathcal{L}\{\cos(at)\} = \dfrac{s}{s^2+a^2}$ then, straightforwardly

$$\mathcal{L}\{-a\sin(at)\} = s\frac{s}{s^2+a^2} - 1 = -\frac{a^2}{s^2+a^2}$$

i.e $\mathcal{L}\{\sin(at)\} = \dfrac{a}{s^2+a^2}$ as expected.

2. Using Theorem 2.1 gives

$$\mathcal{L}\left\{\frac{\sin t}{t}\right\} = s\mathcal{L}\left\{\int_0^t \frac{\sin u}{u}du\right\}$$

In the text (after Theorem 2.3) we have derived that

$$\mathcal{L}\left\{\int_0^t \frac{\sin u}{u} du\right\} = \frac{1}{s}\tan^{-1}\left\{\frac{1}{s}\right\},$$

in fact this calculation is that one in reverse. The result

$$\mathcal{L}\left\{\frac{\sin t}{t}\right\} = \tan^{-1}\left\{\frac{1}{s}\right\}$$

is immediate. In order to derive the required result, the following manipulations need to take place:

$$\mathcal{L}\left\{\frac{\sin t}{t}\right\} = \int_0^\infty e^{-st}\frac{\sin t}{t}dt$$

and if we substitute $ua = t$ the integral becomes

$$\int_0^\infty e^{-asu}\frac{\sin(au)}{u}du.$$

This is still equal to $\tan^{-1}\left\{\frac{1}{s}\right\}$. Writing $p = as$ then gives the result. (p is a dummy variable of course that can be re-labelled s.)

3. The calculation is as follows:

$$\mathcal{L}\left\{\int_0^t p(v)dv\right\} = \frac{1}{s}\mathcal{L}\{p(v)\}$$

so

$$\mathcal{L}\left\{\int_0^t \int_0^v F(u)du dv\right\} = \frac{1}{s}\mathcal{L}\left\{\int_0^v F(u)du\right\} = \frac{1}{s^2}f(s)$$

as required.

4. Using Theorem 2.3 we get

$$\mathcal{L}\left\{\int_0^t \frac{\cos(au)-\cos(bu)}{u}du\right\} = \frac{1}{s}\int_s^\infty \frac{u}{a^2+u^2} - \frac{u}{b^2+u^2}du.$$

These integrals are standard "ln" and the result $\frac{1}{s}\ln\left(\frac{s^2+a^2}{s^2+b^2}\right)$ follows at once.

5. This transform is computed directly as follows

$$\mathcal{L}\left\{\frac{2\sin t\sinh t}{t}\right\} = \mathcal{L}\left\{\frac{e^t\sin t}{t}\right\} - \mathcal{L}\left\{\frac{e^{-t}\sin t}{t}\right\}.$$

Using the first shift theorem (Theorem 1.2) and the result of Exercise 2 above yields the result that the required Laplace transform is equal to

$$\tan^{-1}\left(\frac{1}{s-1}\right) - \tan^{-1}\left(\frac{1}{s+1}\right) = \tan^{-1}\left(\frac{2}{s^2}\right).$$

(The identity $\tan^{-1}(x) - \tan^{-1}(y) = \tan^{-1}\left(\frac{x-y}{1+xy}\right)$ has been used.)

6. This follows straight from the definition of Laplace transform:

$$\lim_{s\to\infty} \bar{f}(s) = \lim_{s\to\infty} \int_0^\infty e^{-st}F(t)dt = \int_0^\infty \lim_{s\to\infty} e^{-st}F(t)dt = 0.$$

It also follows from the final value theorem (Theorem 2.7) in that if $\lim_{s\to\infty} s\bar{f}(s)$ is finite then by necessity $\lim_{s\to\infty} \bar{f}(s) = 0$.

7. These problems are all reasonably straightforward

$$\text{(a)}\ \frac{2(2s+7)}{(s+4)(s+2)} = \frac{3}{s+2} + \frac{1}{s+4}$$

and inverting each Laplace transform term by term gives the result $3e^{-2t} + e^{-4t}$

$$\text{(b) Similarly} \frac{s+9}{s^2-9} = \frac{2}{s-3} - \frac{1}{s+3}$$

and the result of inverting each term gives $2e^{3t} - e^{-3t}$

$$\text{(c)} \frac{s^2+2k^2}{s(s^2+4k^2)} = \frac{1}{2}\left(\frac{1}{s} + \frac{s}{s^2+4k^2}\right)$$

and inverting gives the result

$$\frac{1}{2} + \frac{1}{2}\cos(2kt) = \cos^2(kt).$$

$$\text{(d)}\ \frac{1}{s(s+3)^2} = \frac{1}{9s} - \frac{1}{9(s+3)} - \frac{1}{3(s+3)^2}$$

which inverts to

$$\frac{1}{9} - \frac{1}{9}(3t+1)e^{-3t}.$$

(d) This last part is longer than the others. The partial fraction decomposition is best done by computer algebra, although hand computation is possible. The result is

$$\frac{1}{(s-2)^2(s+3)^3} = \frac{1}{125(s-2)^2} - \frac{3}{625(s-2)} + \frac{1}{25(s+3)^3} + \frac{2}{125(s+3)^2} + \frac{3}{625(s+3)}$$

and the inversion gives $\dfrac{e^{2t}}{625}(5t-3) + \dfrac{e^{-3t}}{1250}(25t^2+20t+6)$.

8. (a) $F(t) = 2 + \cos(t) \to 3$ as $t \to 0$, and as $\dfrac{2}{s} + \dfrac{s}{s^2+1}$ we also have that $sf(s) \to 2+1 = 3$ as $s \to \infty$ hence verifying the initial value theorem.
(b) $F(t) = (4+t)^2 \to 16$ as $t \to 0$. In order to find the Laplace transform, we expand and evaluate term by term so that $sf(s) = 16 + 8/s + 2/s^2$ which obviously also tends to 16 as $s \to \infty$ hence verifying the theorem once more.
9. (a) $F(t) = 3 + e^{-t} \to 3$ as $t \to \infty$. $f(s) = \dfrac{3}{s} + \dfrac{1}{s+1}$ so that $sf(s) \to 3$ as $s \to 0$ as required by the final value theorem.
(b) With $F(t) = t^3e^{-t}$, we have $f(s) = 6/(s+1)^4$ and as $F(t) \to 0$ as $t \to \infty$ and $sf(s)$ also tends to the limit 0 as $s \to 0$ the final value theorem is verified.
10. For small enough t, we have that

$$\sin(\sqrt{t}) = \sqrt{t} + O(t^{3/2})$$

and using the standard form (Appendix B):

$$\mathcal{L}\{t^{x-1}\} = \frac{\Gamma\{x\}}{s^x}$$

with $x = 3/2$ gives

$$\mathcal{L}\{\sin(\sqrt{t})\} = \mathcal{L}\{\sqrt{t}\} + \cdots = \frac{\Gamma\{3/2\}}{s^{3/2}} + \cdots$$

and using that $\Gamma\{3/2\} = (1/2)\Gamma\{1/2\} = \sqrt{\pi}/2$ we deduce that

$$\mathcal{L}\{\sin(\sqrt{t})\} = \frac{\sqrt{\pi}}{2s^{3/2}} + \cdots .$$

Also, using the formula given,

$$\frac{k}{s^{3/2}}e^{-\frac{1}{4s}} = \frac{k}{s^{3/2}} + \cdots .$$

Comparing these series for large values of s, equating coefficients of $s^{-3/2}$ gives

$$k = \frac{\sqrt{\pi}}{2}.$$

11. Using the power series expansions for sin and cos gives

$$\sin(t^2) = \sum_{n=0}^{\infty}(-1)^n \frac{t^{4n+2}}{(2n+1)!}$$

and

$$\cos(t^2) = \sum_{n=0}^{\infty}(-1)^n \frac{t^{4n}}{2n!}.$$

Taking the Laplace transform term by term gives

$$\mathcal{L}\{\sin(t^2)\} = \sum_{n=0}^{\infty}(-1)^n \frac{(4n+2)!}{(2n+1)!s^{4n+3}}$$

and

$$\mathcal{L}\{\cos(t^2)\} = \sum_{n=0}^{\infty}(-1)^n \frac{(4n)!}{(2n)!s^{4n+1}}.$$

12. Given that $Q(s)$ is a polynomial with n distinct zeros, we may write

$$\frac{P(s)}{Q(s)} = \frac{A_1}{s-a_1} + \frac{A_2}{s-a_2} + \cdots + \frac{A_k}{s-a_k} + \cdots + \frac{A_n}{s-a_n}$$

where the A_ks are some real constants to be determined. Multiplying both sides by $s - a_k$ then letting $s \to a_k$ gives

$$A_k = \lim_{s \to a_k} \frac{P(s)}{Q(s)}(s-a_k) = P(a_k) \lim_{s \to a_k} \frac{(s-a_k)}{Q(s)}.$$

Using l'Hôpital's rule now gives

$$A_k = \frac{P(a_k)}{Q'(a_k)}$$

for all $k = 1, 2, \ldots, n$. This is true for all k, thus we have established that

$$\frac{P(s)}{Q(s)} = \frac{P(a_1)}{Q'(a_1)} \frac{1}{(s-a_1)} + \cdots + \frac{P(a_k)}{Q'(a_k)} \frac{1}{(s-a_k)} + \cdots \frac{P(a_n)}{Q'(a_n)} \frac{1}{(s-a_n)}.$$

Taking the inverse Laplace transform gives the result

$$\mathcal{L}^{-1}\left\{\frac{P(s)}{Q(s)}\right\} = \sum_{k=1}^{n} \frac{P(a_k)}{Q'(a_k)} e^{a_k t}$$

sometimes known as Heaviside's expansion formula.

13. All of the problems in this question are solved by evaluating the Laplace transform explicitly.

$$\text{(a) } \mathcal{L}\{H(t-a)\} = \int_a^\infty e^{-st}dt = \frac{e^{-as}}{s}.$$

$$\text{(b) } \mathcal{L}\{f_1(t)\} = \int_0^2 (t+1)e^{-st}dt + \int_2^\infty 3e^{-st}dt.$$

Evaluating the right-hand integrals gives the solution

$$\frac{1}{s} + \frac{1}{s^2}(e^{-2s}-1).$$

$$\text{(c) } \mathcal{L}\{f_2(t)\} = \int_0^2 (t+1)e^{-st}dt + \int_2^\infty 6e^{-st}dt.$$

Once again, evaluating gives

$$\frac{1}{s} + \frac{3}{s}e^{-2s} + \frac{1}{s^2}(e^{-2s}-1)$$

(d) As the function $f_1(t)$ is in fact continuous in the interval $[0, \infty)$ the formula for the derivative of the Laplace transform (Theorem 2.1) can be used to give the result $\frac{1}{s}(e^{-2s}-1)$ at once. Alternatively, f_1 can be differentiated (it is $1 - H(t-2)$) and evaluated directly.

14. We use the formula for the Laplace transform of a periodic function Theorem 2.8 to give

$$\mathcal{L}\{F(t)\} = \frac{\int_0^{2c} e^{-st}F(t)dt}{(1-e^{-2sc})}.$$

The numerator is evaluated directly:

$$\int_0^{2c} e^{-st}F(t)dt = \int_0^c te^{-st}dt + \int_c^{2c}(2c-t)e^{-st}dt$$

which after routine integration by parts simplifies to

$$\frac{1}{s^2}(e^{-sc}-1)^2.$$

The Laplace transform is thus

$$\mathcal{L}\{F(t)\} = \frac{1}{1-e^{-2sc}}\frac{1}{s^2}(e^{-sc}-1)^2 = \frac{1}{s^2}\frac{1-e^{-sc}}{1+e^{-sc}}$$

which simplifies to

$$\frac{1}{s^2}\tanh\left(\frac{1}{2}sc\right).$$

15. Evaluating the integral gives:

$$f(s) = \int_0^\infty e^{-st}F(t)dt = \int_a^{a+h}\frac{e^{-st}}{h}dt = \frac{e^{-as}}{sh}(1-e^{-sh})$$

so

$$f(s) = \frac{e^{-as}}{sh}(1-e^{-sh}).$$

If we let $h \to 0$ then $F(t) \to \delta(t-a)$ and we get

$$f(s) = \frac{e^{-sa}}{s}\lim_{h\to 0}\left\{\frac{1-e^{-sh}}{h}\right\} = \frac{e^{-sa}}{s}s = e^{-as}$$

using L'Hôpital's Rule, so

$$\mathcal{L}\{\delta(t-a)\} = e^{-as}.$$

Exercises 3.6

1. (a) If we substitute $u = t - \tau$ into the definition of convolution then

$$g * f = \int_0^t g(\tau)f(t-\tau)d\tau$$

becomes

$$-\int_t^0 g(u-\tau)f(u)du = g * f.$$

(b) Associativity is proved by effecting the transformation $(u, \tau) \to (x, y)$ where $u = t - x - y$, and $\tau = y$ on the expression

$$f * (g * h) = \int_0^t \int_0^{t-\tau} f(\tau)g(u)h(t-\tau-u)dud\tau.$$

The area covered by the double integral does not change under this transformation, it remains the right-angled triangle with vertices $(0, t)$, $(0, 0)$ and $(t, 0)$. The calculation proceeds as follows:

$$dud\tau = \frac{\partial(u,\tau)}{\partial(x,y)}dxdy = -dxdy$$

so that

$$f * (g * h) = \int_0^t \int_0^{t-x} f(y)g(t-x-y)h(x)dydx$$
$$= \int_0^t h(x)\left[\int_0^{t-x} f(y)g(t-x-y)dy\right]dx$$
$$= \int_0^t h(x)[f * g](t-x)dx = h * (f * g)$$

and this is $(f * g) * h$ by part (a) which establishes the result.
(c) Taking the Laplace transform of the expression $f * f^{-1} = 1$ gives

$$\mathcal{L}\{f\} \cdot \mathcal{L}\{f^{-1}\} = \frac{1}{s}$$

from which

$$\mathcal{L}\{f^{-1}\} = \frac{1}{s\bar{f}(s)}$$

using the usual notation ($\bar{f}(s)$ is the Laplace transform of $f(t)$). It must be the case that $\dfrac{1}{s\bar{f}(s)} \to 0$ as $s \to \infty$. The function f^{-1} is not uniquely defined. Using the properties of the Dirac-δ function, we can also write

$$\int_0^{t+} f(\tau)\delta(t-\tau)d\tau = f(t)$$

from which

$$f^{-1}(t) = \frac{\delta(t-\tau)}{f(t)}.$$

Clearly, $f(t) \neq 0$.

2. Since $\mathcal{L}\{f\} = \bar{f}$ and $\mathcal{L}\{1\} = 1/s$ we have

$$\mathcal{L}\{f * 1\} = \frac{\bar{f}}{s}$$

so that, on inverting

$$\mathcal{L}^{-1}\left\{\frac{\bar{f}}{s}\right\} = f * 1 = \int_0^t f(\tau)d\tau$$

as required.

3. These convolution integrals are straightforward to evaluate:

$$\text{(a)} \quad t * \cos t = \int_0^t (t - \tau) \cos \tau d\tau$$

this is, using integration by parts

$$1 - \cos t.$$

$$\text{(b)} \quad t * t = \int_0^t (t - \tau)\tau d\tau = \frac{t^3}{6}.$$

$$\text{(c)} \quad \sin t * \sin t = \int_0^t \sin(t - \tau) \sin \tau d\tau = \frac{1}{2} \int_0^t [\cos(2\tau - t) - \cos t] d\tau$$

this is now straightforwardly

$$\frac{1}{2}(\sin t - t \cos t).$$

$$\text{(d)} \quad e^t * t = \int_0^t e^{t-\tau} \tau d\tau$$

which on integration by parts gives

$$-1 - t + e^{-t}.$$

$$\text{(e)} \quad e^t * \cos t = \int_0^t e^{t-\tau} \cos \tau d\tau.$$

Integration by parts twice yields the following equation

$$\int_0^t e^{t-\tau} \cos \tau d\tau = \left[e^{-\tau} \sin \tau - e^{-\tau} \cos \tau\right]_0^t - \int_0^t e^{t-\tau} \cos \tau d\tau$$

from which

$$\int_0^t e^{t-\tau} \cos \tau d\tau = \frac{1}{2}(\sin t - \cos t + e^t).$$

4. (a) This is proved by using l'Hôpital's rule as follows

$$\lim_{x \to 0} \left\{ \frac{\operatorname{erf}(x)}{x} \right\} = \lim_{x \to 0} \frac{1}{x} \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt = \frac{2}{\sqrt{\pi}} \frac{d}{dx} \int_0^x e^{-t^2} dt$$

and using Leibnitz' rule (or differentiation under the integral sign) this is

$$\lim_{x\to 0}\frac{2}{\sqrt{\pi}}e^{-x^2}=\frac{2}{\sqrt{\pi}}$$

as required.

(b) This part is tackled using power series expansions. First note that

$$e^{-x^2}=1-x^2+\frac{x^4}{2!}-\frac{x^6}{3!}+\cdots+(-1)^{n+1}\frac{x^{2n}}{n!}+\cdots.$$

Integrating term by term (uniformly convergent for all x) gives

$$\int_0^{\sqrt{t}}e^{-x^2}dx=t^{1/2}-\frac{t^{3/2}}{3}+\frac{t^{5/2}}{5.2!}-\frac{t^{7/2}}{7.3!}+\cdots+(-1)^{n+1}\frac{t^{n+1/2}}{(2n+1)\cdot n!}+\cdots$$

from which

$$t^{-\frac{1}{2}}\operatorname{erf}(\sqrt{t})=\frac{2}{\sqrt{\pi}}\left(1-\frac{t}{3}+\frac{t^2}{5.2!}-\frac{t^3}{7.3!}+\cdots\right.$$
$$\left.+(-1)^{n+1}\frac{t^n}{(2n+1)\cdot n!}+\cdots\right).$$

Taking the Laplace transform of this series term by term (again justified by the uniform convergence of the series for all t) gives

$$\mathcal{L}^{-1}\{t^{-\frac{1}{2}}\operatorname{erf}(\sqrt{t})\}=\frac{2}{\sqrt{\pi}}\left(\frac{1}{s}-\frac{1}{3s^2}+\frac{1}{5s^3}-\frac{1}{7s^4}+\cdots\right.$$
$$\left.+\frac{(-1)^n}{(2n+1)s^{n+1}}+\cdots\right)$$

and taking out a factor $1/\sqrt{s}$ leaves the arctan series for $1/\sqrt{s}$. Hence we get the required result:

$$\mathcal{L}^{-1}\{t^{-1/2}\operatorname{erf}(\sqrt{t})\}=\frac{2}{\sqrt{\pi s}}\tan^{-1}\left(\frac{1}{\sqrt{s}}\right).$$

5. All of these differential equations are solved by taking Laplace transforms. Only some of the more important steps are shown.
(a) The transformed equation is

$$s\bar{x}(s)-x(0)+3\bar{x}(s)=\frac{1}{s-2}$$

from which, after partial fractions,

$$\bar{x}(s) = \frac{1}{s+3} + \frac{1}{(s-2)(s+3)} = \frac{4/5}{s+3} + \frac{1/5}{s-2}.$$

Inverting gives

$$x(t) = \frac{4}{5}e^{-3t} + \frac{1}{5}e^{2t}.$$

(b) This equation has Laplace transform

$$(s+3)\bar{x}(s) - x(0) = \frac{1}{s^2+1}$$

from which

$$\bar{x}(s) = \frac{x(0)}{s+3} - \frac{1/10}{s+3} + \frac{s/10 - 3/10}{s^2+1}.$$

The boundary condition $x(\pi) = 1$ is not natural for Laplace transforms, however inverting the above gives

$$x(t) = \left(x(0) - \frac{1}{10}\right)e^{-3t} - \frac{1}{10}\cos(t) + \frac{3}{10}\sin(t)$$

and this is 1 when $x = \pi$, from which

$$x(0) - \frac{1}{10} = \frac{9}{10}e^{3\pi}$$

and the solution is

$$x(t) = \frac{9}{10}e^{3(\pi-t)} - \frac{1}{10}\cos(t) + \frac{3}{10}\sin(t).$$

(c) This equation is second order; the principle is the same but the algebra is messier. The Laplace transform of the equation is

$$s^2\bar{x}(s) + 4s\bar{x}(s) + 5\bar{x}(s) = \frac{8}{s^2+1}$$

and rearranging using partial fractions gives

$$\bar{x}(s) = \frac{s+2}{(s+2)^2+1} + \frac{1}{(s+2)^2+1} - \frac{s}{s^2+1} + \frac{1}{s^2+1}.$$

Taking the inverse then yields the result

$$x(t) = e^{-2t}(\cos t + \sin t) + \sin t - \cos t.$$

(d) The Laplace transform of the equation is

$$(s^2 - 3s - 2)\bar{x}(s) - s - 1 + 3 = \frac{6}{s}$$

from which, after rearranging and using partial fractions,

$$\bar{x}(s) = -\frac{3}{s} + \frac{4(s - \frac{3}{2})}{(s - \frac{3}{2})^2 - \frac{17}{4}} - \frac{5}{(s - \frac{3}{2})^2 - \frac{17}{4}}$$

which gives the solution

$$x(t) = -3 + 4e^{\frac{3}{2}t} \cosh\left(\frac{t}{2}\sqrt{17}\right) - \frac{10}{\sqrt{17}} e^{\frac{3}{2}t} \sinh\left(\frac{t}{2}\sqrt{17}\right).$$

(e) This equation is solved in a similar way. The transformed equation is

$$s^2\bar{y}(s) - 3s + \bar{y}(s) - 1 = \frac{6}{s^2 + 4}$$

from which

$$\bar{y}(s) = -\frac{2}{s^2 + 4} + \frac{3s + 3}{s^2 + 1}$$

and inverting, the solution

$$y(t) = -\sin(2t) + 3\cos t + 3\sin t$$

results.

6. Simultaneous ODEs are transformed into simultaneous algebraic equations and the algebra to solve them is often horrid. For parts (a) and (c) the algebra can be done by hand, for part (b) computer algebra is almost compulsory.
(a) The simultaneous equations in the transformed state after applying the boundary conditions are

$$(s - 2)\bar{x}(s) - (s + 1)\bar{y}(s) = \frac{6}{s - 3} + 3$$
$$(2s - 3)\bar{x}(s) + (s - 3)\bar{y}(s) = \frac{6}{s - 3} + 6$$

from which we solve and rearrange to obtain

$$\bar{x}(s) = \frac{4}{(s - 3)(s - 1)} + \frac{3s - 1}{(s - 1)^2}$$

so that, using partial fractions

$$\bar{x}(s) = \frac{2}{s-3} + \frac{1}{s-1} + \frac{2}{(s-1)^2}$$

giving, on inversion

$$x(t) = 2e^{3t} + e^t + 2te^t.$$

In order to find $y(t)$ we eliminate dy/dt from the original pair of simultaneous ODEs to give

$$y(t) = -3e^{3t} - \frac{5}{4}x(t) + \frac{3}{4}\frac{dx}{dt}.$$

Substituting for $x(t)$ then gives

$$y(t) = -e^{3t} + e^t - te^t.$$

(b) This equation is most easily tackled by substituting the derivative of

$$y = -4\frac{dx}{dt} - 6x + 2\sin(2t)$$

into the second equation to give

$$5\frac{d^2}{dx^2} + 6\frac{dx}{dt} + x = 4\cos(2t) + 3e^{-2t}.$$

The Laplace transform of this is then

$$5(s^2\bar{x}(s) - sx(0) - x'(0)) + 6(s\bar{x}(s) - x(0)) + \bar{x}(s) = \frac{4s}{s^2+4} + \frac{3}{s+2}.$$

After inserting the given boundary conditions and rearranging we are thus face with inverting

$$\bar{x}(s) = \frac{10s+2}{5s^2+6s+1} + \frac{4s}{(s^2+4)(5s^2+6s+1)} + \frac{3}{(s+2)(5s^2+6s+1)}.$$

Using a partial fractions package gives

$$\bar{x}(s) = \frac{29}{20(s+1)} + \frac{1}{3(s+2)} + \frac{2225}{1212(5s+1)} - \frac{4(19s-24)}{505(s^2+4)}$$

and inverting yields

$$x(t) = \frac{1}{3}e^{-2t} + \frac{29}{20}e^{-t} + \frac{445}{1212}e^{-\frac{1}{5}t} - \frac{76}{505}\cos(2t) + \frac{48}{505}\sin(2t).$$

Substituting back for $y(t)$ gives

$$y(t) = \frac{2}{3}e^{2t} - \frac{29}{10}e^{-t} - \frac{1157}{606}e^{-\frac{1}{5}t} + \frac{72}{505}\cos(2t) + \frac{118}{505}\sin(2t).$$

(c) This last problem is fully fourth order, but we do not change the line of approach. The Laplace transform gives the simultaneous equations

$$(s^2 - 1)\bar{x}(s) + 5s\bar{y}(s) - 5 = \frac{1}{s^2}$$
$$-2s\bar{x}(s) + (s^2 - 4)\bar{y}(s) - s = -\frac{2}{s}$$

in which the boundary conditions have already been applied. Solving for $\bar{y}(s)$ gives

$$\bar{y}(s) = \frac{s^4 + 7s^2 + 4}{s(s^2 + 4)(s^2 + 1)} = \frac{1}{s} - \frac{2}{3}\frac{s}{s^2 + 4} + \frac{2}{3}\frac{s}{s^2 + 1}$$

which inverts to the solution

$$y(t) = 1 - \frac{2}{3}\cos(2t) + \frac{2}{3}\cos t.$$

Substituting back into the second original equation gives

$$x(t) = -t - \frac{5}{3}\sin t + \frac{4}{3}\sin(2t).$$

7. Using Laplace transforms, the transform of x is given by

$$\bar{x}(s) = \frac{A}{(s^2 + 1)(s^2 + k^2)} + \frac{v_0}{(s^2 + k^2)} + \frac{sx(0)}{(s^2 + k^2)}.$$

If $k \neq 1$ this inverts to

$$x(t) = \frac{A}{k^2 - 1}\left(\sin t - \frac{\sin(kt)}{k}\right) + \frac{v_0}{k}\sin(kt) + x_0\cos(kt).$$

If $k = 1$ there is a term $(1 + s^2)^2$ in the denominator, and the inversion can be done using convolution. The result is

$$x(t) = \frac{A}{2}(\sin t - t\cos t) + v_0 \sin t + x(0)\cos t$$

and it can be seen that this tends to infinity as $t \to \infty$ due to the term $t\cos t$. This is called a *secular* term. It is not present in the solution for $k \neq 1$ which is purely oscillatory. The presence of a secular term denotes resonance.

8. Taking the Laplace transform of the equation, using the boundary conditions and rearranging gives

$$\bar{x}(s) = \frac{sv_0 + g}{s^2(s+a)}$$

which after partial fractions becomes

$$\bar{x}(s) = \frac{-\frac{1}{a^2}(av_0 - g)}{s+a} + \frac{-\frac{1}{a^2}(av_0 - g)s + \frac{g}{a}}{s^2}.$$

This inverts to the expression in the question. The speed

$$\frac{dx}{dt} = \frac{g}{a} - \frac{(av_0 - g)}{a}e^{-at}.$$

As $t \to \infty$ this tends to g/a which is the required terminal speed.

9. The set of equations in matrix form is determined by taking the Laplace transform of each. The resulting algebraic set is expressed in matrix form as follows:

$$\begin{pmatrix} 1 & -1 & -1 & 0 \\ R_1 & sL_2 & 0 & 0 \\ R_1 & 0 & R_3 & 1/C \\ 0 & 0 & 1 & -s \end{pmatrix} \begin{pmatrix} \bar{j}_1 \\ \bar{j}_2 \\ \bar{j}_3 \\ \bar{q}_3 \end{pmatrix} = \begin{pmatrix} 0 \\ L_2 j_2(0) + E\omega/(\omega^2 + s^2) \\ E\omega/(\omega^2 + s^2) \\ -q_3(0) \end{pmatrix}.$$

10. The Laplace transform of this fourth order equation is

$$k(s^4\bar{y}(s) - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0)) = \frac{\omega_0}{c}\left(\frac{c}{s} - \frac{1}{s^2} + \frac{e^{-as}}{s^2}\right)$$

Using the boundary conditions is easy for those given at $x = 0$, the others give

$$y''(0) - 2cy'''(0) + \frac{5}{6}\omega_0 c^2 = 0 \quad \text{and} \quad y'''(0) = \frac{1}{2}\omega_0 c.$$

So $y''(0) = \frac{1}{6}\omega_0 c^2$ and the full solution is, on inversion

$$y(x) = \frac{1}{12}\omega_0 c^2 x^2 - \frac{1}{12}\omega_0 c x^3 + \frac{\omega_0}{120}\left[5cx^4 - x^5 + (x-c)^5 H(x-c)\right]$$

where $0 \le x \le 2c$. Differentiating twice and putting $x = c/2$ gives $y''(c/2) = \frac{1}{48}\omega_0 c^2$.

11. Taking the Laplace transform and using the convolution theorem gives

$$\bar{\phi}(s) = \frac{2}{s^3} + \bar{\phi}(s)\frac{1}{s^2+1}$$

from which

$$\bar{\phi}(s) = \frac{2}{s^5} + \frac{2}{s^2}.$$

Inversion gives the solution

$$\phi(t) = t^2 + \frac{1}{12}t^4.$$

Exercises 4.7

1. The Riemann–Lebesgue lemma is stated as Theorem 4.2. As the constants b_n in a Fourier sine series for $g(t)$ in $[0, \pi]$ are given by

$$b_n = \frac{2}{\pi}\int_0^{\pi} g(t)\sin(nt)dt$$

and these sine functions form a basis for the linear space of piecewise continuous functions in $[0, \pi]$ (with the usual inner product) of which $g(t)$ is a member, the Riemann–Lebesgue lemma thus immediately gives the result. More directly, Parseval's Theorem:

$$\int_{-\pi}^{\pi}[g(t)]^2 dt = \pi a_0^2 + \pi\sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

yields the results

$$\lim_{n\to\infty}\int_{-\pi}^{\pi} g(t)\cos(nt)dt = 0$$

$$\lim_{n\to\infty}\int_{-\pi}^{\pi} g(t)\sin(nt)dt = 0$$

as the nth term of the series on the right has to tend to zero as $n \to \infty$. As $g(t)$ is piecewise continuous over the half range $[0, \pi]$ and is free to be defined as odd over the full range $[-\pi, \pi]$, the result follows.

2. The Fourier series is found using the formulae in Sect. 4.2. The calculation is routine if lengthy and the answer is

$$\begin{aligned} f(t) = \frac{5\pi}{16} &- \frac{2}{\pi}\left(\cos(t) + \frac{\cos(3t)}{3^2} + \frac{\cos(5t)}{5^2} + \cdots\right) \\ &- \frac{2}{\pi}\left(\frac{\cos(2t)}{2^2} + \frac{\cos(6t)}{6^2} + \frac{\cos(10t)}{10^2}\cdots\right) \\ &+ \frac{1}{\pi}\left(\sin(t) - \frac{\sin(3t)}{3^2} + \frac{\sin(5t)}{5^2} - \frac{\sin(7t)}{7^2}\cdots\right). \end{aligned}$$

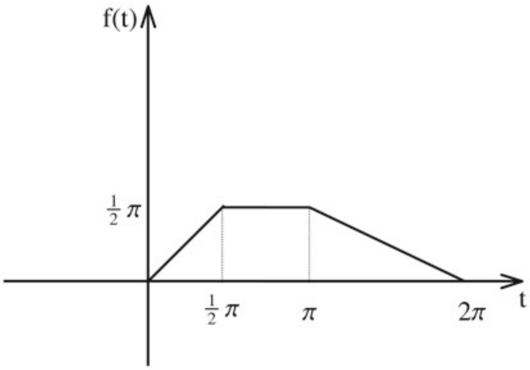


Fig. A.1 The original function composed of *straight line* segments

This function is displayed in Fig. A.1.

3. The Fourier series for $H(x)$ is found straightforwardly as

$$\frac{1}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\sin(2n-1)x}{2n-1}.$$

Put $x = \pi/2$ and we get the series in the question and its sum:

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}$$

a series attributed to the Scottish mathematician James Gregory (1638–1675).

4. The Fourier series has the value

$$f(x) \sim -\frac{8}{\pi} \sum_{n=1}^{\infty} \frac{n \sin(2nx)}{(2n+1)(2n-1)}.$$

5. This is another example where the Fourier series is found straightforwardly using integration by parts. The result is

$$1 - x^2 \sim \frac{1}{2}\left(\pi - \frac{\pi^3}{3}\right) - 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos(nx).$$

As the Fourier series is in fact continuous for this example there is no controversy, at $x = \pi$, $f(x) = 1 - \pi^2$.

6. Evaluating the integrals takes a little stamina this time.

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} e^{ax} \sin(nx) dx$$

and integrating twice by parts gives

$$b_n = \left[\frac{1}{a\pi}\sin(nx) - \frac{n}{a^2\pi}\cos(nx)\right]_{-\pi}^{\pi} - \frac{n^2}{a^2}b_n$$

from which

$$b_n = -\frac{2n\sinh(a\pi(-1)^n)}{a^2\pi}, \quad n = 1,\ 2, \ldots.$$

Similarly,

$$a_n = \frac{2a\sinh(a\pi(-1)^n)}{a^2\pi}, \quad n = 1,\ 2, \ldots,$$

and

$$a_0 = \frac{2\sinh(\pi a)}{\pi a}.$$

This gives the series in the question. Putting $x = 0$ gives the equation

$$e^0 = 1 = \frac{\sinh(\pi a)}{\pi}\left\{\frac{1}{a} + 2\sum_{n=1}^{\infty}\frac{(-1)^n a}{n^2 + a^2}\right\}$$

from which

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2 + a^2} = \frac{1}{2a^2}(a\pi\operatorname{cosech}(a\pi) - 1).$$

Also, since

$$\sum_{-\infty}^{\infty}\frac{(-1)^n}{n^2 + a^2} = 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2 + a^2} + \frac{1}{a^2},$$

we get the result

$$\sum_{-\infty}^{\infty}\frac{(-1)^n}{n^2 + a^2} = \frac{\pi}{a}\operatorname{cosech}(a\pi).$$

Putting $x = \pi$ and using Dirichlet's theorem (Theorem 4.3) we get

$$\frac{1}{2}(f(\pi) + f(-\pi)) = \cosh(a\pi) = \frac{\sinh(\pi a)}{\pi}\left\{\frac{1}{a} + 2\sum_{n=1}^{\infty}\frac{a}{n^2 + a^2}\right\}$$

from which

$$\sum_{n=1}^{\infty}\frac{1}{n^2 + a^2} = \frac{1}{2a^2}(a\pi\coth(a\pi) - 1).$$

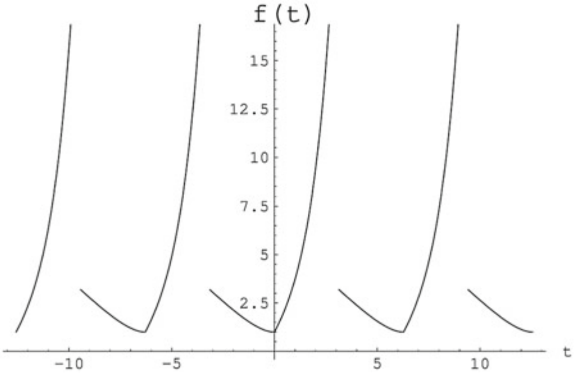


Fig. A.2 The function $f(t)$

Also, since

$$\sum_{-\infty}^{\infty}\frac{1}{n^2+a^2}=2\sum_{n=1}^{\infty}\frac{1}{n^2+a^2}+\frac{1}{a^2}$$

we get the result

$$\sum_{-\infty}^{\infty}\frac{1}{n^2+a^2}=\frac{\pi}{a}\coth(a\pi).$$

7. The graph is shown in Fig. A.2 and the Fourier series itself is given by

$$\begin{aligned}f(t)=\frac{1}{2}\pi+\frac{1}{\pi}\sinh(\pi)&\\ &+\frac{2}{\pi}\sum_{n=1}^{\infty}\left[\frac{(-1)^n-1}{n^2}+\frac{(-1)^n\sinh(\pi)}{n^2+1}\right]\cos(nt)\\ &-\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2+1}\sinh(\pi)\sin(nt).\end{aligned}$$

8. The Fourier series expansion over the range $[-\pi,\pi]$ is found by integration to be

$$f(t)=\frac{2}{3}\pi^2+\sum_{n=1}^{\infty}\left[\frac{2}{n^2}\cos(nt)+\frac{(-1)^n}{n}\pi\sin(nt)\right]-\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\sin(2n-1)t}{(2n-1)^3}$$

and Fig. A.3 gives a picture of it. The required series are found by first putting $t=0$ which gives

$$\pi^2 = \frac{2}{3}\pi^2 + 2\sum_{n=1}^{\infty} \frac{1}{n^2}$$

from which

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Putting $t = \pi$ gives, using Dirichlet's theorem (Theorem 4.3)

$$\frac{\pi^2}{2} = \frac{2}{3}\pi^2 - 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$$

from which

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}.$$

9. The sine series is given by the formula

$$a_n = 0 \quad b_n = \frac{2}{\pi}\int_0^{\pi} (t-\pi)^2 \sin(nt)dt$$

with the result

$$f(t) \sim \frac{8}{\pi}\sum_{k=1}^{\infty} \frac{\sin(2k-1)t}{2k+1} + 2\pi\sum_{n=1}^{\infty} \frac{\sin(nt)}{n}.$$

This is shown in Fig. A.5. The cosine series is given by

$$b_n = 0 \quad a_n = \frac{2}{\pi}\int_0^{\pi} (t-\pi)^2 \cos(nt)dt$$

from which

$$f(t) \sim -\frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{\cos(nt)}{n^2}$$

and this is pictured in Fig. A.4.

10. Parseval's theorem (Theorem 4.8) is

$$\int_{-\pi}^{\pi} [f(t)]^2 dt = \pi a_0^2 + \pi\sum_{n=1}^{\infty} (a_n^2 + b_n^2).$$

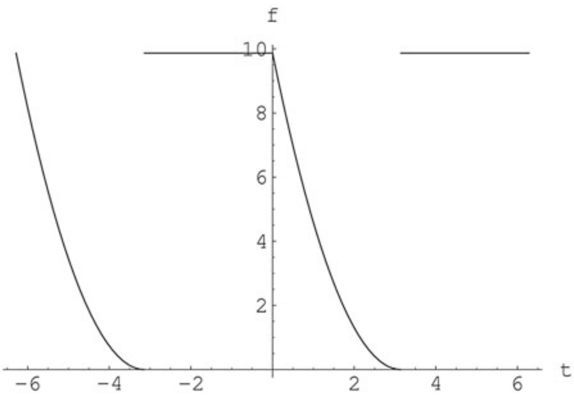


Fig. A.3 The function $f(t)$ as a full Fourier series

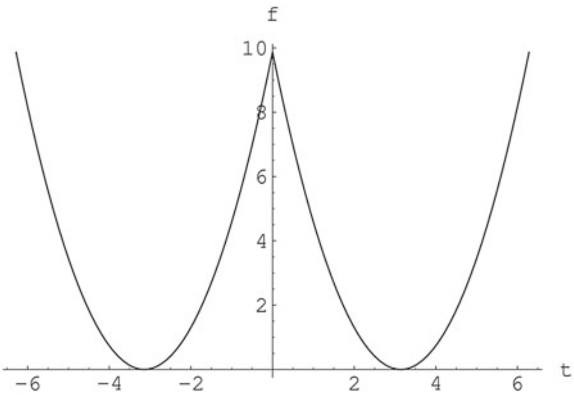


Fig. A.4 The function $f(t)$ as an even function

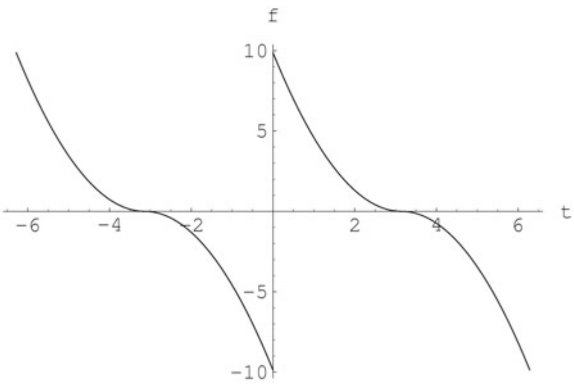


Fig. A.5 The function $f(t)$ as an odd function

Applying this to the Fourier series in the question is not straightforward, as we need the version for sine series. This is easily derived as

$$\int_0^{\pi} [f(t)]^2 dt = \frac{\pi}{2} \sum_{n=1}^{\infty} b_n^2.$$

The left hand side is

$$\int_0^{\pi} t^2(\pi - t)^2 dt = \frac{\pi^5}{30}.$$

The right hand side is

$$\frac{32}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^6}$$

whence the result

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^6} = \frac{\pi^6}{960}.$$

Noting that

$$\sum_{n=1}^{\infty} \frac{1}{n^6} = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^6} + \frac{1}{2^6} \sum_{n=1}^{\infty} \frac{1}{n^6}$$

gives the result

$$\sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{64}{63} \frac{\pi^6}{960} = \frac{\pi^6}{945}.$$

11. The Fourier series for the function x^4 is found as usual by evaluating the integrals

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x^4 \cos(nx) dx$$

and

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x^4 \sin(nx) dx.$$

However as x^4 is an even function, $b_n = 0$ and there are no sine terms. Evaluating the integral for the cosine terms gives the series in the question. With $x = 0$, the series becomes

$$0 = \frac{\pi^4}{5} + 8\pi^2 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} - 48 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4}$$

and using the value of the second series $(\pi^2)/12$ gives

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4} = \frac{7\pi^4}{15}\frac{1}{48} = \frac{7\pi^4}{720}.$$

Differentiating term by term yields the Fourier series for x^3 for $-\pi < x < \pi$ as

$$x^3 \sim \sum_{n=1}^{\infty} (-1)^n \frac{2}{n^3}(6 - \pi^2 n^2)\sin(nx).$$

12. Integrating the series

$$x \sim \sum_{n=1}^{\infty} \frac{2}{n}(-1)^{n+1}\sin(nx)$$

term by term gives

$$\frac{x^2}{2} \sim \sum_{n=1}^{\infty} \frac{2}{n^2}(-1)^n \cos(nx) + A$$

where A is a constant of integration. Integrating both sides with respect to x between the limits $-\pi$ and π gives

$$\frac{\pi^3}{3} = 2A\pi.$$

Hence $A = \pi^2/6$ and the Fourier series for x^2 is

$$x^2 \sim \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2}\cos(nx),$$

where $-\pi < x < \pi$. Starting with the Fourier series for x^4 over the same range and integrating term by term we get

$$\frac{x^5}{5} \sim \frac{\pi^4 x}{5} + \sum_{n=1}^{\infty} \frac{8(-1)^n}{n^5}(\pi^2 n^2 - 6)\sin(nx) + B$$

where B is the constant of integration. This time setting $x = 0$ immediately gives $B = 0$, but there is an x on the right-hand side that has to be expressed in terms of a Fourier series. We thus use

$$x \sim \sum_{n=1}^{\infty} \frac{2}{n}(-1)^{n+1} \sin(nx)$$

to give the Fourier series for x^5 in $[-\pi, \pi]$ as

$$x^5 \sim \sum_{n=1}^{\infty}(-1)^n \left[\frac{40\pi^2}{n^3} - \frac{240}{n^5} - \frac{2\pi^4}{n}\right] \sin(nx).$$

13. Using the complex form of the Fourier series, we have that

$$V(t) = \sum_{n=-\infty}^{\infty} c_n e^{2n\pi it/5}.$$

The coefficients are given by the formula

$$c_n = \frac{1}{5}\int_0^5 V(t)e^{2in\pi t/5}dt.$$

By direct calculation they are

$$\begin{aligned}
c_{-4} &= \frac{5}{\pi}\left(i + e^{7\pi i/10}\right)\\
c_{-3} &= \frac{20}{3\pi}\left(i - e^{9\pi i/10}\right)\\
c_{-2} &= \frac{10}{\pi}\left(i - e^{\pi i/10}\right)\\
c_{-1} &= \frac{20}{\pi}\left(i + e^{3\pi i/10}\right)\\
c_0 &= 16\\
c_1 &= \frac{20}{\pi}\left(-i - e^{7\pi i/10}\right)\\
c_2 &= \frac{10}{\pi}\left(-i + e^{9\pi i/10}\right)\\
c_3 &= \frac{20}{3\pi}\left(-i + e^{\pi i/10}\right)\\
c_4 &= \frac{10}{\pi}\left(-i - e^{3\pi i/10}\right).
\end{aligned}$$

14. The differential equation can be written

$$\frac{d}{dt}\left[(1-t^2)\frac{dP_n}{dt}\right] = -n(n+1)P_n.$$

This means that the integral can be manipulated using integration by parts as follows:

$$
\begin{aligned}
\int_{-1}^{1} P_m P_n dt &= -\frac{1}{n(n+1)} \int_{-1}^{1} \frac{d}{dt}\left[(1-t^2)\frac{dP_n}{dt}\right] P_m dt \\
&= -\frac{1}{n(n+1)} \left[(1-t^2)\frac{dP_n}{dt}\frac{dP_m}{dt}\right]_{-1}^{1} \\
&\quad + \frac{1}{n(n+1)} \int_{-1}^{1} \frac{d}{dt}\left[(1-t^2)\frac{dP_m}{dt}\right] P_n dt \\
&= \frac{1}{n(n+1)} \left[(1-t^2)\frac{dP_m}{dt}\frac{dP_n}{dt}\right]_{-1}^{1} + \frac{m(m+1)}{n(n+1)} \int_{-1}^{1} P_m P_n dt \\
&= \frac{m(m+1)}{n(n+1)} \int_{-1}^{1} P_m P_n dt,
\end{aligned}
$$

all the integrated bits being zero. Therefore

$$\int_{-1}^{1} P_m P_n dt = 0$$

unless $m = n$ as required.

15. The first part is straightforward. Substituting $y = \sin(n \ln t)$ into the equation for y gives

$$\frac{dy}{dt} = \frac{n}{t}\cos(n \ln t), \quad \text{so} \quad t\frac{dy}{dt} = n\cos(n \ln t)$$

thus

$$t\frac{d}{dt}\left\{t\frac{dy}{dt}\right\} = -n^2 \sin(n \ln t) = -n^2 y$$

as required. In order to do the next part we follow the last example in the chapter, or the previous exercise and integrate the ordinary differential equation. Write the ODE as:

$$\frac{d}{dt}\left\{t\frac{d\phi_n}{dt}\right\} + \frac{n^2}{t}\phi_n = 0 \tag{A.1}$$

write it again substituting m for n:

$$\frac{d}{dt}\left\{t\frac{d\phi_m}{dt}\right\} + \frac{m^2}{t}\phi_m = 0 \tag{A.2}$$

then form the combination $\phi_m \times$ Eq. (A.1) $- \phi_n \times$ Eq. (A.2) to obtain

$$\phi_m \frac{d}{dt}\left\{t\frac{d\phi_n}{dt}\right\} - \phi_n \frac{d}{dt}\left\{t\frac{d\phi_m}{dt}\right\} + \frac{n^2-m^2}{t}\phi_m\phi_n = 0$$

Integrating this between the two zeros (these are where $\ln t$ is 0 and π so $t = 1$ and e^π). The first pair of terms yield to integration by parts:

$$\left[\phi_m \cdot t \cdot \frac{d\phi_n}{dt} - \phi_n \cdot t \cdot \frac{d\phi_m}{dt}\right]_0^{e^\pi} - \int_0^{e^\pi} \frac{d\phi_m}{dt} \cdot t \cdot \frac{d\phi_n}{dt} - \frac{d\phi_n}{dt} \cdot t \cdot \frac{d\phi_m}{dt} dt$$

and both of these are zero, the first term as it vanishes at both limits and in the second the integrand is identically zero. Hence we have

$$\int_1^{e^\pi} \frac{n^2-m^2}{t}\phi_m\phi_n dt = (m^2-n^2)\int_1^{e^\pi} \frac{\phi_m\phi_n}{t} dt = 0$$

which implies orthogonality with weight function $1/t$.

16. The half range Fourier series for $g(x)$ is

$$\sum_{n=1}^{\infty} b_n \sin(nx)$$

where

$$b_n = \frac{2}{\pi}\int_0^\pi \sin(nx)dx = \frac{2}{n\pi}\left[-\cos(nx)\right]_0^\pi = \frac{2}{n\pi}[1-(-1)^n]$$

so $b_{2k} = 0$ and $b_{2k+1} = \dfrac{4}{(2k+1)\pi}$ So with $x = \ln t$

$$f(t) = \frac{4}{\pi}\sum_{k=0}^{\infty} \frac{1}{4k+1} \sin\{(2k+1)\ln t\}$$

Exercises 5.6

1. Using the separation of variable technique with

$$\phi(x,t) = \sum_k X_k(x)T_k(t)$$

gives the equations for $X_k(x)$ and $T_k(t)$ as

$$\frac{T_k'}{T_k} = \frac{\kappa X_k''}{X_k} = -\alpha^2$$

where $-\alpha^2$ is the separation constant. It is negative because $T_k(t)$ must not grow with t. In order to satisfy the boundary conditions we express $x(\pi/4 - x)$ as a Fourier sine series in the interval $0 \le x \le \pi/4$ as follows

$$x(x - \frac{\pi}{4}) = \frac{1}{2\pi}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^3}\sin[4(2k-1)x].$$

Now the function $X_k(x)$ is identified as

$$X_k(x) = \frac{1}{2\pi}\frac{1}{(2k-1)^3}\sin[4(2k-1)x]$$

so that the equation obeyed by $X_k(x)$ together with the boundary conditions $\phi(0,t) = \phi(\pi/4,t) = 0$ for all time are satisfied. Thus

$$\alpha = \frac{4(2k-1)\pi}{\sqrt{\kappa}}.$$

Putting this expression for $X_k(x)$ together with

$$T_k(t) = e^{-16(2k-1)^2\pi^2 t/\kappa}$$

gives the solution

$$\phi(x,t) = \frac{1}{2\pi}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^3}e^{-16(2k-1)^2\pi^2 t/\kappa}\sin[4(2k-1)x].$$

2. The equation is

$$a\frac{\partial^2\phi}{\partial x^2} - b\frac{\partial\phi}{\partial x} - \frac{\partial\phi}{\partial t} = 0.$$

Taking the Laplace transform (in t) gives the ODE

$$a\bar{\phi}'' - b\bar{\phi}' - s\bar{\phi} = 0$$

after applying the boundary condition $\phi(x,0) = 0$. Solving this and noting that

$$\phi(0,t) = 1 \Rightarrow \bar{\phi}(0,s) = \frac{1}{s} \quad \text{and} \quad \bar{\phi}(x,s) \to 0 \quad \text{as} \quad x \to \infty$$

gives the solution

$$\bar{\phi} = \frac{1}{s}\exp\left\{\frac{x}{2a}[b - \sqrt{b^2 + 4as}]\right\}.$$

3. Taking the Laplace transform of the heat conduction equation results in the ODE

$$\kappa \frac{d^2\bar{\phi}}{dx^2} - s\bar{\phi} = -x\left(\frac{\pi}{4} - x\right).$$

Although this is a second order non-homogeneous ODE, it is amenable to standard complimentary function, particular integral techniques. The general solution is

$$\bar{\phi}(x, s) = A\cosh\left(x\sqrt{\frac{s}{\kappa}}\right) + B\sinh\left(x\sqrt{\frac{s}{\kappa}}\right) - \frac{x^2}{s} + \frac{\pi x}{4s} - \frac{2\kappa}{s^2}.$$

The inverse of this gives the expression in the question. If this inverse is evaluated term by term using the series in Appendix A, the answer of Exercise 1 is not regained immediately. However, if the factor $2\kappa t$ is expressed as a Fourier series, then the series solution is the same as that in Exercise 1.

4. Taking the Laplace transform in y gives the ODE

$$\frac{d^2\bar{\phi}}{dx^2} = s\bar{\phi}.$$

Solving this, and applying the boundary condition $\phi(0, y) = 1$ which transforms to

$$\bar{\phi}(0, s) = \frac{1}{s}$$

gives the solution

$$\bar{\phi}(x, s) = \frac{1}{s}e^{-x\sqrt{s}}$$

which inverts to

$$\phi(x, y) = \operatorname{erfc}\left\{\frac{x}{2\sqrt{y}}\right\}.$$

5. Using the transformation suggested in the question gives the equation obeyed by $\phi(x, t)$ as

$$\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = \frac{\partial^2\phi}{\partial x^2}.$$

This is the standard wave equation. To solve this using Laplace transform techniques, we transform in the variable t to obtain the equation

$$\frac{d^2\bar{\phi}}{dx^2} - \frac{s^2}{c^2}\bar{\phi} = -\frac{s}{c^2}\cos(mx).$$

The boundary conditions for $\phi(x, t)$ are

$$\phi(x, 0) = \cos(mx) \quad \text{and} \quad \phi'(x, 0) = -\frac{k}{2}\phi(x, 0) = -\frac{k}{2}\cos(mx).$$

This last condition arises since

$$u'(x,t) = \frac{k}{2}e^{kt/2}\phi(x,t) + e^{kt/2}\phi'(x,t).$$

Applying these conditions gives, after some algebra

$$\bar{\phi}(x,s) = \left[\frac{1}{s} - \frac{s-\frac{k}{2}}{s^2+m^2c^2}\cos(mx)\right]e^{-\frac{sx}{c}} + \frac{s-\frac{k}{2}}{s^2+m^2c^2}\cos(mx).$$

Using the second shift theorem (Theorem 2.4) we invert this to obtain

$$u = \begin{cases} e^{kt/2}(1-\cos(mct-mx)\cos(mx) + \frac{k}{2mc}\sin(mct-mx)\cos(mx) \\ \qquad +\cos(mct)\cos(mx) - \frac{k}{2mc}\sin(mct)\cos(mx)) & t > x/c \\ \\ e^{kt/2}(\cos(mct)\cos(mx) - \frac{k}{2mc}\sin(mct)\cos(mx)) & t < x/c. \end{cases}$$

6. Taking the Laplace transform of the one dimensional heat conduction equation gives

$$s\bar{u} = c^2\bar{u}_{xx}$$

as $u(x,0) = 0$. Solving this with the given boundary condition gives

$$\bar{u}(x,s) = \bar{f}(s)e^{-x\sqrt{s}/c}.$$

Using the standard form

$$\mathcal{L}^{-1}\{e^{-a\sqrt{s}}\} = \frac{a}{2\sqrt{\pi t^3}}e^{-a^2/4t}$$

gives, using the convolution theorem

$$u = \frac{x}{2}\int_0^t f(\tau)\sqrt{\frac{k}{\pi(t-\tau)^3}}e^{-x^2/4k(t-\tau)}d\tau.$$

When $f(t) = \delta(t)$, $u = \frac{x}{2}\sqrt{\frac{k}{\pi t^3}}e^{-x^2/4kt}$.

7. Assuming that the heat conduction equation applies gives that

$$\frac{\partial T}{\partial t} = \kappa\frac{\partial^2 T}{\partial x^2}$$

so that when transformed $\bar{T}(x,s)$ obeys the equation

$$s\bar{T}(x,s) - T(x,0) = \kappa \frac{d^2\bar{T}}{dx^2}(x,s).$$

Now, $T(x,0) = 0$ so this equation has solution

$$\bar{T}(x,s) = \bar{T}_0 e^{-x\sqrt{\frac{s}{\kappa}}}$$

and since

$$\frac{d\bar{T}}{dx}(s,0) = -\frac{\alpha}{s}$$

$$\bar{T}_0 = \alpha\sqrt{\frac{\kappa}{s}}$$

and the solution is using standard forms (or Chap. 3)

$$T(x,t) = \alpha\sqrt{\frac{\kappa}{\pi t}} e^{-x^2/4\kappa t}.$$

This gives, at $x = 0$ the desired form for $T(0,t)$. Note that for non-zero x the solution is not singular at $t = 0$.

8. Taking the Laplace transform of the wave equation given yields

$$s^2\bar{u} - \frac{\partial u}{\partial t}(x,0) = c^2 \frac{d^2\bar{u}}{dx^2}$$

so that substituting the series

$$\bar{u} = \sum_{n=0}^{\infty} \frac{a_n(x)}{s^{n+k+1}}, \quad k \text{ integer}$$

as in Example 5.5 gives $k = 1$ all the odd powers are zero and

$$\frac{a_0}{s^2} + \frac{a_2}{s^4} + \frac{a_6}{s^6} + \cdots - \cos(x) = c^2\left(\frac{a_0''}{s^2} + \frac{a_2''}{s^4} + \frac{a_6''}{s^6} + \cdots\right)$$

so that

$$a_0 = \cos(x), \quad a_2 = c^2 a_0'' = -c^2\cos(x) \quad a_4 = c^4\cos x \quad \text{etc.}$$

Hence

$$\bar{u}(x,s) = \cos(x)\left(\frac{1}{s^2} - \frac{c^2}{s^4} + \frac{c^4}{s^6} - \frac{c^6}{s^8} + \cdots\right)$$

which inverts term by term to

$$u(x,t) = \cos x \left(t - \frac{c^2 t^3}{3!} + \frac{c^4 t^5}{5!} + \cdots \right)$$

which in this case converges to the closed form solution

$$u = \frac{1}{c} \cos x \sin(ct).$$

9. For this problem we proceed similarly. Laplace transforms are taken and the equation to solve this time is

$$s^2 \bar{u} - su(x,0) = c^2 \frac{d^2 \bar{u}}{dx^2}.$$

Once more substituting the series

$$\bar{u} = \sum_{n=0}^{\infty} \frac{a_n(x)}{s^{n+k+1}}, \quad k \text{ integer}$$

gives this time

$$a_0 + \frac{a_1}{s} + \frac{a_2}{s^2} + \cdots - \cos x = c^2 \left(a_0'' + \frac{a_1''}{s} + \frac{a_2''}{s^2} + \cdots \right)$$

so that

$$a_0 = \cos x, \quad a_1 = 0 \;\; a_2 = c^2 a_0'' = -c^2 \cos x \;\; a_3 = 0 \;\; a_4 = c^4 \cos x \;\; \text{etc.}$$

giving

$$\bar{u}(x,s) = \sum_{n=0}^{\infty} \frac{c^{2n}(-1)^n \cos x}{s^{2n+1}}.$$

Inverting term by term gives the answer

$$u(x,t) = \cos x \sum_{n=0}^{\infty} (-1)^n \frac{c^{2n} t^{2n}}{2n!}$$

which in fact in this instance converges to the result

$$u = \cos x \cos(ct).$$

Exercises 6.6

1. With the function $f(t)$ as defined, simple integration reveals that

$$\begin{aligned}F(\omega) &= \int_{-T}^{0}(t+T)e^{-i\omega t}dt + \int_0^T (T-t)e^{-i\omega t}dt\\ &= 2\int_0^T (T-t)\cos(\omega t)dt\\ &= \frac{2}{\omega}\left[(T-t)\frac{\sin(\omega t)}{\omega}\right]_0^T + \frac{2}{\omega}\int_0^T \sin(\omega t)dt\\ &= \frac{2}{\omega}\left[-\frac{\cos(\omega t)}{\omega}\right]_0^T\\ &= \frac{2(1-\cos(\omega T))}{\omega^2}\end{aligned}$$

2. With $f(t) = e^{-t^2}$ the Fourier transform is

$$F(\omega) = \int_{-\infty}^{\infty} e^{-t^2}e^{-i\omega t}dt = \int_{-\infty}^{\infty} e^{-(t-\frac{1}{2}i\omega)^2}e^{-\frac{1}{4}\omega^2}dt.$$

Now although there is a complex number $(\frac{1}{2}i\omega)$ in the integrand, the change of variable $u = t - \frac{1}{2}i\omega$ can still be made. The limits are actually changed to $-\infty - \frac{1}{2}i\omega$ and $\infty - \frac{1}{2}i\omega$ but this does not change its value so we have that

$$\int_{-\infty}^{\infty} e^{-(t-\frac{1}{2}i\omega)^2}dt = \sqrt{\pi}.$$

Hence

$$F(\omega) = \sqrt{\pi}e^{-\frac{1}{4}\omega^2}.$$

3. Consider the Fourier transform of the square wave, Example 6.1. The inverse yields:

$$\frac{A}{\pi}\int_{-\infty}^{\infty}\frac{\sin(\omega T)}{\omega}e^{i\omega t}d\omega = A$$

provided $|t| \le T$. Let $t = 0$ and we get

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin(\omega T)}{\omega}d\omega = 1$$

Putting $T = 1$ and spotting that the integrand is even gives the result.

4. Using the definition of convolution given in the question, we have

$$
\begin{aligned}
f(at) * f(bt) &= \int_{-\infty}^{\infty} f(a(t-\tau)) f(b\tau) d\tau. \\
&= e^{-at} \int_{0}^{\infty} f(b\tau - a\tau) d\tau \\
&= -e^{-at} \frac{1}{b-a} [f(bt-at) - 1] \\
&= \frac{f(at) - f(bt)}{b-a}.
\end{aligned}
$$

As $b \to a$ we have

$$
\begin{aligned}
f(at) * f(at) &= -\lim_{b\to a} \frac{f(bt) - f(at)}{b-a} \\
&= -\frac{d}{da} f(at) = -tf'(at) = tf(at).
\end{aligned}
$$

5. With

$$
g(x) = \int_{-\infty}^{\infty} f(t) e^{-2\pi i x t} dt
$$

let $u = t - 1/2x$, so that $du = dt$. This gives

$$
e^{-2\pi i x t} = e^{-2\pi i x(u+1/2x)} = -e^{-2\pi x u}.
$$

Adding these two versions of $g(x)$ gives

$$
\begin{aligned}
|g(x)| &= \left| \frac{1}{2} \int_{-\infty}^{\infty} (f(u) - f(u+1/2x)) e^{-2\pi i x u} du \right| \\
&\le \frac{1}{2} \int_{-\infty}^{\infty} |f(u) - f(u+1/2x)| du
\end{aligned}
$$

and as $x \to \infty$, the right hand side $\to 0$. Hence

$$
\int_{-\infty}^{\infty} f(t) \cos(2\pi x t) dt \to 0 \ \text{ and } \ \int_{-\infty}^{\infty} f(t) \sin(2\pi x t) dt \to 0
$$

which illustrates the Riemann–Lebesgue lemma.

6. First of all we note that

$$
G(it) = \int_{-\infty}^{\infty} g(\omega) e^{-i^2 \omega t} d\omega = \int_{-\infty}^{\infty} g(\omega) e^{\omega t} d\omega
$$

therefore

$$\int_{-\infty}^{\infty} f(t)G(it)dt = \int_{-\infty}^{\infty} f(t) \int_{-\infty}^{\infty} g(\omega)e^{\omega t} d\omega dt.$$

Assuming that the integrals are uniformly convergent so that their order can be interchanged, the right hand side can be written

$$\int_{-\infty}^{\infty} g(\omega) \int_{-\infty}^{\infty} f(t)e^{\omega t} dt d\omega,$$

which is, straight away, in the required form

$$\int_{-\infty}^{\infty} g(\omega)F(i\omega)d\omega.$$

Changing the dummy variable from ω to t completes the proof.

7. Putting $f(t) = 1 - t^2$ where $f(t)$ is zero outside the range $-1 \le t \le 1$ and $g(t) = e^{-t}, \quad 0 \le t < \infty$, we have

$$F(\omega) = \int_{-1}^{1} (1 - t^2)e^{-i\omega t} dt$$

and

$$G(\omega) = \int_{0}^{\infty} e^{-t} e^{i\omega t} dt.$$

Evaluating these integrals (the first involves integrating by parts twice) gives

$$F(\omega) = \frac{4}{\omega^3} (\omega \cos \omega - \sin \omega)$$

and

$$G(\omega) = \frac{1}{1 + i\omega}.$$

Thus, using Parseval's formula from the previous question, the imaginary unit disappears and we are left with

$$\int_{-1}^{1} (1 + t)dt = \int_{0}^{\infty} \frac{4e^{-t}}{t^3} (t \cosh t - \sinh t)\, dt$$

from which the desired integral is 2. Using Parseval's theorem

$$\int_{-\infty}^{\infty} |f(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(\omega)|^2 d\omega$$

we have that

$$\int_{-1}^{1}(1-t^2)^2 dt = \frac{1}{2\pi}\int_0^\infty \frac{16\,(t\cos t - \sin t)^2}{t^6}dt.$$

Evaluating the integral on the left we get

$$\int_0^\infty \frac{(t\cos t - \sin t)^2}{t^6}dt = \frac{\pi}{15}.$$

8. The ordinary differential equation obeyed by the Laplace transform $\bar{u}(x, s)$ is

$$\frac{d^2\bar{u}(x, s)}{dx^2} - \frac{s}{k}\bar{u}(x, s) = -\frac{g(x)}{k}.$$

Taking the Fourier transform of this we obtain the solution

$$\bar{u}(x, s) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{G(\omega)}{s+\omega^2 k}e^{i\omega x}d\omega$$

where

$$G(\omega) = \int_{-\infty}^{\infty} g(x)e^{-i\omega x}dx$$

is the Fourier transform of $g(x)$. Now it is possible to write down the solution by inverting $\bar{u}(x, s)$ as the Laplace variable s only occurs in the denominator of a simple fraction. Inverting using standard forms thus gives

$$u(x, t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega)e^{i\omega}e^{-\omega^2 kt}d\omega.$$

It is possible by completing the square and using the kind of "tricks" seen in Sect. 3.2 to convert this into the solution that can be obtained directly by Laplace transforms and convolution, namely

$$u(x, t) = \frac{1}{2\sqrt{\pi t}}\int_{-\infty}^{\infty} e^{-(x-\tau)^2/4t}g(\tau)d\tau.$$

9. To convert the partial differential equation into the integral form is a straightforward application of the theory of Sect. 6.4. Taking Fourier transforms in x and y using the notation

$$v(\lambda, y) = \int_{-\infty}^{\infty} u(x, y)e^{-i\lambda x}dx$$

and

$$w(\lambda, \mu) = \int_{-\infty}^{\infty} v(\lambda, y)e^{-i\mu y}dy$$

we obtain

$$-\lambda^2 w - \mu^2 w + k^2 w = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi,\eta)e^{-\lambda\xi}e^{-\mu\eta}d\xi d\eta.$$

Using the inverse Fourier transform gives the answer in the question. The conditions are that for both $u(x, y)$ and $f(x, y)$ all first partial derivatives must vanish at $\pm\infty$.

10. The Fourier series written in complex form is

$$f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

where

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx.$$

Now it is straightforward to use the window function $W(x)$ to write

$$c_n = \frac{1}{2}\int_{-\infty}^{\infty} W\left(\frac{x-\pi}{2\pi}\right) f(x)e^{-inx}dx.$$

11. The easiest way to see how discrete Fourier transforms are derived is to consider a sampled time series as the original time series $f(t)$ multiplied by a function that picks out the discrete (sampled) values leaving all other values zero. This function is related to the Shah function (train of Dirac-δ functions) is not necessarily (but is usually) equally spaced. It is designed by using the window function $W(x)$ met in the last question. With such a function, taking the Fourier transform results in the finite sum of the kind seen in the question. The inverse is a similar evaluation, noting that because of the discrete nature of the function, there is a division by the total number of data points.
12. Inserting the values $\{1, 2, 1\}$ into the series of the previous question, $N = 3$ and $T = 1$ so we get the three values

$$F_0 = 1 + 2 + 1 = 4; \quad F_1 = 1 + 2e^{-2\pi i/3} + e^{-4\pi i/3} = e^{-2\pi i/3};$$

and

$$F_2 = 1 + 2e^{-4\pi i/3} + e^{-8\pi i/3} = e^{-4\pi i/3}.$$

13. Using the definition and essential property of the Dirac-δ function we have

$$F(\omega) = \int_{-\infty}^{\infty} \delta(t - t_0)e^{-i\omega t}dt = e^{-i\omega t_0}$$

inverting this gives the required answer

$$\delta(t - t_0) = \int_{-\infty}^{\infty} e^{-i\omega t_0} e^{i\omega t} d\omega = \int_{-\infty}^{\infty} e^{i\omega(t-t_0)} d\omega.$$

whence the second result is

$$\delta(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega t} d\omega.$$

14. Using the exponential forms of sine and cosine the Fourier transforms are immediately

$$\mathcal{F}\{\cos(\omega_0 t)\} = \frac{1}{2}(\delta(\omega - \omega_0) + \delta(\omega + \omega_0))$$
$$\mathcal{F}\{\sin(\omega_0 t)\} = \frac{1}{2i}(\delta(\omega - \omega_0) - \delta(\omega + \omega_0)).$$

Exercises 7.9

1. The wavelets being considered are $[1, 1, 1, 1]^T$, $[1, 1, -1, -1]^T$, $[1, -1, 0, 0]^T$ and $[0, 0, 1, -1]^T$ So we solve α times the first plus β times the second plus γ time the third plus δ times the fourth equals $[4, 8, 10, 14]^T$ which leads to

$$\alpha + \beta + \gamma = 4$$
$$\alpha + \beta - \gamma = 8$$
$$\alpha - \beta + \delta = 10$$
$$\alpha - \beta - \delta = 14.$$

Solving (easily by hand) gives $\alpha = 9$, $\beta = -3$, $\gamma = 6$ and $\delta = 12$. When the right hand side is $[a, b, c, d]^T$ the result is still easy to obtain by hand:

$$\alpha = \frac{1}{4}(a + b + c + d)$$
$$\beta = \frac{1}{4}(a + b - c - d)$$
$$\gamma = \frac{1}{2}(a + b)$$
$$\delta = \frac{1}{2}(c + d)$$

2. The 16 basis functions are the columns of the matrix:

$$
\begin{pmatrix}
1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
1 & -1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & -1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
1 & -1 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & -1 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
1 & -1 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
1 & -1 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
1 & -1 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & -1 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1
\end{pmatrix}
$$

This layout is useful for the next part of the question. It is quickly apparent that each column is orthogonal to any other as the inner product is found by multiplying equivalent terms and adding up the whole 16. All of these sums are zero. The vectors are not orthonormal. Finally the vector $(1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)^T$ can be put in terms of the basis functions as follows. Labelling the basis vectors a, b, up to p, then forming $\frac{1}{2}(a + b)$ we get a vector with 8 ones then below this, 8 zeros. Adding this to the third column c and diving by 2 gives a vector with 4 ones in the first four places and twelve zeros below that. The calculation is $\frac{1}{4}(a + b) + \frac{1}{2}c$. Now continuing in this fashion, add this vector to the vector represented by column e and divide by two to give two ones in the first two entries with 14 zeros below. The calculation is $\frac{1}{8}(a + b) + \frac{1}{4}c + \frac{1}{2}e$. Finally add this to the vector represented by i and divide by 2 and we get $(1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)^T$ that achieves the required vector. The calculation is

$$\frac{1}{16}(a + b) + \frac{1}{8}c + \frac{1}{4}e + \frac{1}{2}i$$

3. Calculating the Fourier transform of $\psi_{j,k}(t) = 2^{j/2}\psi\left(2^j t - k\right)$ is a bit tricky. Note that

$$\psi(t) = \begin{cases} 1 & 0 \le t < \frac{1}{2} \\ -1 & \frac{1}{2} \le t < 1 \\ 0 & \text{otherwise} \end{cases}$$

so $\psi\left(2^j t - k\right)$ is always either 1 −1 or 0. In fact

$$\psi\left(2^j t - k\right) = \begin{cases} 1 & 2^j k \le t < (k + \frac{1}{2})2^j \\ -1 & (k + \frac{1}{2})2^j \le t < (k + 1)2^j \\ 0 & \text{otherwise} \end{cases}$$

So, using the definition of Fourier transform:

$$\hat{\psi}_{j,k}(\omega) = \int_{-\infty}^{\infty} \psi_{j,k}(t) e^{-i\omega t} dt = 2^{j/2} \int_{-\infty}^{\infty} \psi\left(2^j t - k\right) e^{-i\omega t} dt$$

then using the above definition of $\psi\left(2^j t - k\right)$ we get

$$\begin{aligned}
\hat{\psi}_{j,k}(\omega) &= \int_{-\infty}^{\infty} 2^{j/2}\psi\left(2^j t - k\right) e^{-i\omega t} dt \\
&= 2^{j/2}\left[\int_{k2^{-j}}^{(k+\frac{1}{2})2^{-j}} e^{-i\omega t} dt - \int_{(k+\frac{1}{2})2^{-j}}^{(k+1)2^{-j}} e^{-i\omega t} dt\right] \\
&= 2^{j/2}\left[-\frac{e^{-i\omega t}}{i\omega}\Big|_{k2^{j/2}}^{(k+\frac{1}{2})2^{j/2}} + \frac{e^{-i\omega t}}{i\omega}\Big|_{(k+\frac{1}{2})2^{j/2}}^{(k+1)2^{j/2}}\right] \\
&= 2^{j/2}\frac{e^{-i\omega k 2^{-j}}}{i\omega}\left\{1 - 2e^{-i\omega(k+\frac{1}{2})2^{-j}} + e^{-i\omega(k+1)2^{-j}}\right\}
\end{aligned}$$

Some manipulation then gives

$$\hat{\psi}_{j,k}(\omega) = -\frac{2^{(j+4)/2}}{i\omega} e^{-i\omega(k+\frac{1}{2})2^{-j}} \sin^2\left(\frac{\omega}{4}2^{-j}\right)$$

which is the required Fourier transform.

4. Consider the general offspring (daughter) wavelet function

$$2^{j/2}\psi\left(2^j t - k\right)$$

So that

$$\int_{-\infty}^{\infty} |2^{j/2}\psi\left(2^j t - k\right)|^2 dt = 2^j \int_{k2^{-j}}^{(k+1)2^{-j}} 1 dt = 2^j[(k+1)2^{-j} - k2^{-j}] = 1.$$

This is independent of both j and k and so is the same as the mother wavelet $j = k = 0$ and for all daughter wavelets. This proves the result. It has the value 1.

5. With $f(t) = e^{-\frac{1}{2}t^2}$ before we calculate either the centre μ_f or the RMS value, we need to find the norm $||f(t)||$ defined by

$$||f(t)||^2 = \int_{-\infty}^{\infty} |f(t)|^2 dt = \int_{-\infty}^{\infty} e^{-t^2} dt = \sqrt{\pi}$$

so $||f(t)|| = \pi^{1/4}$. The centre μ_f is defined by

$$\mu_f = \frac{1}{||f(t)||^2}\int_{-\infty}^{\infty} t|e^{-\frac{1}{2}t^2}|^2 dt = \pi^{-1/2}\left[-\frac{1}{2}e^{-t^2}\right]_{-\infty}^{\infty} = 0$$

so the centre is zero, this should not be a surprise. It could be deduced at once from the symmetry of $f(t)$.

$$\Delta_f = \frac{1}{||f(t)||}\left[\int_{-\infty}^{\infty} t^2 e^{-t^2} dt\right]^{1/2}$$

Using

$$\int_{-\infty}^{\infty} t^2 e^{-t^2} dt = \left[t \cdot \frac{1}{2}e^{-t^2}\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \frac{1}{2}e^{-t^2} dt = \frac{1}{2}\sqrt{\pi}$$

gives

$$\Delta_f = \frac{1}{\pi^{1/4}}\sqrt{\frac{1}{2}\sqrt{\pi}} = \frac{1}{\sqrt{2}}$$

Given $\hat{f}(\omega) = \sqrt{2\pi}e^{-\frac{1}{2}\omega^2}$ the calculation of the centre $\mu_{\hat{f}}$ and $\Delta_{\hat{f}}$ follows along the same lines. This time $||\hat{f}(\omega)||^2 = 2\pi\sqrt{\pi}$, the centre $\mu_{\hat{f}}$ is of course once again zero. The RMS $\Delta_{\hat{f}}$ is once again $\frac{1}{\sqrt{2}}$ as the factor 2π cancels. Hence the Heisenberg inequality becomes

$$\Delta_f \cdot \Delta_{\hat{f}} = \frac{1}{2}$$

that is, equality.

6. Repeating the details of the above calculation is not necessary. With

$$f(t) = \frac{1}{2\sqrt{\alpha\pi}}e^{-\frac{t^2}{4\alpha}}$$

the norm is $||f(t)||$ where

$$||f(t)||^2 = \frac{1}{4\alpha\pi}\int_{-\infty}^{\infty} e^{-\frac{t^2}{2\alpha}} dt = \frac{1}{2\sqrt{2\alpha\pi}}.$$

Thus the centre $\mu_f = 0$ as before but the RMS Δ_f is given by

$$\Delta_f^2 = \frac{1}{||f(t)||^2}\int_{-\infty}^{\infty} t^2 e^{-\frac{t^2}{2\alpha}} dt = \alpha.$$

Similarly, $||\hat{f}(\omega)||$ is given by

$$||\hat{f}(\omega)||^2 = \int_{-\infty}^{\infty} e^{-2\alpha\omega^2} d\omega = \sqrt{\frac{\pi}{2\alpha}}$$

with $\mu_{\hat{f}} = 0$ and

$$\Delta_{\hat{f}}^2 = \frac{1}{||\hat{f}(\omega)||^2} \int_{-\infty}^{\infty} \omega^2 e^{-2\alpha\omega^2} d\omega = \frac{1}{4\alpha}.$$

Thus giving

$$\Delta_f \cdot \Delta_{\hat{f}} = \frac{1}{2}$$

as before, independent of α.

7. Linearity is straightforward to show. Form the linear sum

$$\alpha f_{1,G}(t_0, \omega) + \beta f_{2,G}(t_0, \omega) = \alpha \int_{-\infty}^{\infty} f_1(t) \overline{b_{t_0,\omega}(t)} dt + \beta \int_{-\infty}^{\infty} f_2(t) \overline{b_{t_0,\omega}(t)} dt$$

The right hand side can be written

$$\int_{-\infty}^{\infty} (\alpha f_1(t) + \beta f_2) \overline{b_{t_0,\omega}(t)} dt = f_G(t_0, \omega)$$

where

$$f(t) = \alpha f_1(t) + \beta f_2(t).$$

This establishes linearity.

(a) If $f_0 = f(t - t_1)$ then

$$f_{1G_b}(t_1, \omega) = \int_{-\infty}^{\infty} f(t - t_1) b(t - t_0) e^{-i\omega t} dt$$

Write $t - t_1 = \tau$ then $dt = d\tau$ and the right hand side is

$$= \int_{-\infty}^{\infty} f(\tau) b(\tau + t_1 - t_0) e^{-i\omega\tau - i\omega t_1} d\tau = e^{-i\omega t_1} f_{1G_b}(t_0 - t_1, \omega)$$

as required.

(b) With

$$f_2 = f(t) e^{i\omega_2 t}$$

we have

$$f_{2G_b}(t_0, \omega) = \int_{-\infty}^{\infty} f_2(t) e^{i\omega_2 t} b(t - t_0) e^{-i\omega t} dt = \int_{-\infty}^{\infty} f_2(t) b(t - t_0) e^{i(\omega - \omega_2)t} dt$$

and the right hand side is $f_{2G_b}(t_0, \omega - \omega_2)$ as required.

8. This follows example 7.9 in the text. The function $f_b(t) = f(t)b(t)$ and is given by

$$f_b(t) = \begin{cases} (1+t)\sin(\pi t) & -1 \le t < 0 \\ (1-t)\sin(\pi t) & 0 \le t < 1 \\ 0 & \text{otherwise} \end{cases}.$$

Thus

$$f_G(1,\omega) = \int_{-\infty}^{\infty} f_b(t)e^{-i\omega t}dt = \int_{-1}^{0}(1+t)\sin(\pi t)e^{-i\omega t}dt + \int_{0}^{1}(1-t)\sin(\pi t)e^{-i\omega t}dt$$

The two integrals on the right are combined by letting $\tau = -t$ in the first to give

$$f_G(1,\omega) = 2i\int_0^1 (1-t)\sin(\pi t)\sin(\omega t)dt.$$

This yields to integration by parts and the answer is:

$$f_G(1,\omega) = 2i\left[\frac{1-\cos\{(\pi-\omega)t\}}{(\pi-\omega)^2} - \frac{1-\cos\{(\pi+\omega)t\}}{(\pi+\omega)^2}\right]$$

Exercises 8.7

1. In all of these examples, the location of the pole is obvious, and the residue is best found by use of the formula

$$\lim_{z\to a}(z-a)f(z)$$

where $z = a$ is the location of the simple pole. In these answers, the location of the pole is followed after the semicolon by its residue. Where there is more than one pole, the answers are sequential, poles first followed by the corresponding residues.

$$\begin{aligned}
&\text{(i) } z = -1;\quad 1,\\
&\text{(ii) } z = 1;\quad -1,\\
&\text{(iii) } z = 1, 3i, -3i;\quad \frac{1}{2}, \frac{5}{12}(3-i), \frac{5}{12}(3+i),\\
&\text{(iv) } z = 0, -2, -1;\quad \frac{3}{2}, -\frac{5}{2}, 1,\\
&\text{(v) } z = 0;\quad 1,\\
&\text{(vi) } z = n\pi\quad (-1)^n n\pi,\quad n \text{ integer},\\
&\text{(vii) } z = n\pi;\quad (-1)^n e^{n\pi},\ n \text{ integer}.
\end{aligned}$$

2. As in the first example, the location of the poles is straightforward. The methods vary. For parts (i), (ii) and (iii) the formula for finding the residue at a pole of order n is best, viz.

$$\frac{1}{(n-1)!}\lim_{z\to a}\frac{d^{(n-1)}}{dz^{(n-1)}}\{(z-a)^n f(z)\}.$$

For part (iv) expanding both numerator and denominator as power series and picking out the coefficient of $1/z$ works best. The answers are as follows

$$\begin{aligned}
&\text{(i)} \quad z=1, \quad \text{order 2res} = 4\\
&\text{(ii)} \quad z=i, \quad \text{order 2res} = -\frac{1}{4}i\\
&\qquad\;\; z=-i, \text{ order 2res} = \frac{1}{4}i\\
&\text{(iii)} \quad z=0, \quad \text{order 3res} = -\frac{1}{2}\\
&\text{(iv)} \quad z=0, \quad \text{order 2res} = 1.
\end{aligned}$$

3. (i) Using the residue theorem, the integral is $2\pi i$ times the sum of the residues of the integrand at the three poles. The three residues are:

$$\frac{1}{3}(1-i)\;\;(\text{at } z=1),\quad \frac{4}{15}(-2-i)\;\;(\text{at } z=-2),\quad \frac{1}{5}(1+3i)\;\;(\text{at } z=-i).$$

The sum of these times $2\pi i$ gives the result

$$-\frac{2\pi}{15}.$$

(ii) This time the residue (calculated easily using the formula) is 6, whence the integral is $12\pi i$.
(iii) For this integral we use a semi circular contour on the upper half plane. By the estimation lemma, the integral around the curved portion tends to zero as the radius gets very large. Also the integral from $-\infty$ to 0 along the real axis is equal to the integral from 0 to ∞ since the integrand is even. Thus we have

$$2\int_0^\infty \frac{1}{x^6+1}dx = 2\pi i(\text{sum of residues at } z = e^{\pi i/6},\;\; i,\;\; e^{5\pi i/6})$$

from which we get the answer $\dfrac{\pi}{3}$.
(iv) This integral is evaluated using the same contour, and similar arguments tell us that

$$2\int_0^\infty \frac{\cos(2\pi x)}{x^4+x^2+1}dx = 2\pi i(\text{sum of residues at } z = e^{\pi i/3},\quad e^{2\pi i/3}).$$

(Note that the complex function considered is $\dfrac{e^{2\pi i z}}{z^4+z^2+1}$. Note also that the poles of the integrand are those of $z^6 - 1$ but excluding $z = \pm 1$.) The answer is, after a little algebra

$$-\frac{\pi}{2\sqrt{3}}e^{-\pi/\sqrt{3}}.$$

4. Problems (i) and (ii) are done by using the function

$$f(z) = \frac{(\ln(z))^2}{z^4+1}.$$

Integrated around the indented semi circular contour of Fig. 8.8, there are poles at $z = (\pm 1 \pm i)/\sqrt{2}$. Only those at $(\pm 1 + i)/\sqrt{2}$ or $z = e^{\pi i/4},\quad e^{3\pi i/4}$ are inside the contour. Evaluating

$$\int_{C'} f(z)dz$$

along all the parts of the contour gives the following contributions: those along the curved bits eventually contribute nothing (the denominator gets very large in absolute magnitude as the radius of the big semi-circle $\to \infty$, the integral around the small circle $\to 0$ as its radius $r \to 0$ since $r(\ln r)^2 \to 0$.) The contributions along the real axis are

$$\int_0^\infty \frac{(\ln x)^2}{x^4+1}dx$$

along the positive real axis where $z = x$ and

$$\int_0^\infty \frac{(\ln x + i\pi)^2}{x^4+1}dx$$

along the negative real axis where $z = xe^{i\pi}$ so $\ln z = \ln x + i\pi$. The residue theorem thus gives

$$2\int_0^\infty \frac{(\ln x)^2}{x^4+1}dx + 2\pi i\int_0^\infty \frac{\ln x}{x^4+1}dx - \pi^2\int_0^\infty \frac{1}{x^4+1}dx \qquad \text{(A.3)}$$
$$= 2\pi i\{\text{sum of residues}\}.$$

The residue at $z = a$ is given by

$$\frac{(\ln a)^2}{4a^3}$$

using the formula for the residue of a simple pole. These sum to

$$-\frac{\pi^2}{64\sqrt{2}}(8-10i).$$

Equating real and imaginary parts of Eq. (A.3) gives the answers

$$\text{(i)}\quad \int_0^\infty \frac{(\ln x)^2}{x^4+1}dx = \frac{3\pi^3\sqrt{2}}{64};\quad \text{(ii)}\quad \int_0^\infty \frac{\ln x}{x^4+1}dx = -\frac{\pi^2}{16}\sqrt{2}$$

once the result

$$\int_0^\infty \frac{1}{x^4+1}dx = \frac{\pi}{2\sqrt{2}}$$

from Example 8.1(ii) is used.
(iii) The third integral also uses the indented semi circular contour of Fig. 8.8. The contributions from the large and small semi circles are ultimately zero. There is a pole at $z=i$ which has residue $e^{\pi\lambda i/2}/2i$ and the straight parts contribute

$$\int_0^\infty \frac{x^\lambda}{1+x^2}dx$$

(positive real axis), and

$$-\int_\infty^0 \frac{x^\lambda e^{\lambda i\pi}}{1+x^2}dx$$

(negative real axis). Putting the contributions together yields

$$\int_0^\infty \frac{x^\lambda}{1+x^2}dx + e^{\lambda i\pi}\int_0^\infty \frac{x^\lambda}{1+x^2}dx = \pi e^{\lambda i\pi/2}$$

from which

$$\int_0^\infty \frac{x^\lambda}{1+x^2}dx = \frac{\pi}{2\cos(\frac{\lambda\pi}{2})}.$$

5. These inverse Laplace transforms are all evaluated form first principles using the Bromwich contour, although it is possible to deduce some of them by using previously derived results, for example if we assume that

$$\mathcal{L}^{-1}\left\{\frac{1}{\sqrt{s}}\right\} = \frac{1}{\sqrt{\pi t}}$$

then we can carry on using the first shift theorem and convolution. However, we choose to use the Bromwich contour. The first two parts follow closely Example 8.5, though none of the branch points in these problems is in the expo-

nential. The principle is the same.
(i) This Bromwich contour has a cut along the negative real axis from -1 to $-\infty$. It is shown as Fig. A.6. Hence

$$\mathcal{L}^{-1}\left\{\frac{1}{s\sqrt{s+1}}\right\} = \frac{1}{2\pi i}\int_{Br}\frac{e^{st}}{s\sqrt{s+1}}ds.$$

The integral is thus split into the following parts

$$\int_{C'} = \int_{Br} + \int_{\Gamma} + \int_{AB} + \int_{\gamma} + \int_{CD} = 2\pi i(\text{residue at } s=0)$$

where C' is the whole contour, Γ is the outer curved part, AB is the straight portion above the cut ($\Im s > 0$) γ is the small circle surrounding the branch point $s = -1$ and CD is the straight portion below the cut ($\Im s < 0$). The residue is one, the curved parts of the contour contribute nothing in the limit. The important contributions come from the integrals along AB and CD. On AB we can put $s = xe^{i\pi} - 1$. This leads to the integral

$$\int_{AB} = \int_{\infty}^{0}\frac{e^{t(-x-1)}}{(-x-1)i\sqrt{x}}dx$$

On CD we can put $s = xe^{-i\pi} - 1$. The integrals do not cancel because of the square root in the denominator (the reason the cut is there of course!). They in fact exactly reinforce. So the integral is

$$\int_{CD} = \int_{0}^{\infty}\frac{e^{t(-x-1)}}{(-x-1)(-i\sqrt{x})}dx.$$

Hence

$$\int_{C} = \int_{Br} - 2\int_{\infty}^{0}\frac{e^{-t}e^{-xt}}{i(x+1)\sqrt{x}}dx = 2\pi i.$$

Using the integral

$$\int_{0}^{\infty}\frac{e^{-xt}}{(x+1)\sqrt{x}}dx = -\pi e^{t}[-1+\operatorname{erf}\sqrt{t}]$$

gives the answer that

$$\mathcal{L}^{-1}\left\{\frac{1}{s\sqrt{s+1}}\right\} = \frac{1}{2\pi i}\int_{Br}\frac{e^{st}}{s\sqrt{s+1}}ds = \operatorname{erf}\sqrt{t}.$$

(ii) This second part is tackled in a similar way. The contour is identical (Fig. A.6). The important step is the correct parametrisation of the straight parts of the

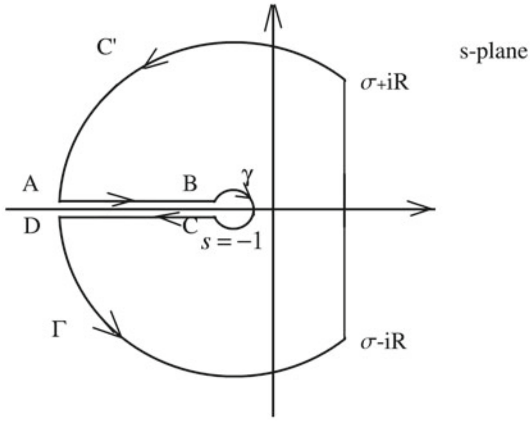


Fig. A.6 The cut Bromwich contour

contour just above and just below the branch cut. This time the two integrals along the cut are

$$\int_{AB} = -\int_{\infty}^{0} \frac{e^{(-x-1)t}}{1+i\sqrt{x}}dx$$

and

$$\int_{CD} = -\int_{0}^{\infty} \frac{e^{(-x-1)t}}{1-i\sqrt{x}}dx,$$

and the integrals combine to give

$$\mathcal{L}^{-1}\left\{\frac{1}{1+\sqrt{s+1}}\right\} = \frac{1}{2\pi i}\int_0^{\infty} \frac{e^{-t}e^{-xt}2i\sqrt{x}}{1+x}dx$$

which gives the result

$$\frac{e^{-t}}{\sqrt{\pi t}} - \operatorname{erfc}\sqrt{t}.$$

(iii) This inverse can be obtained from part (ii) by using the first shift theorem (Theorem 1.2). The result is

$$\mathcal{L}^{-1}\left\{\frac{1}{1+\sqrt{s}}\right\} = \frac{1}{\sqrt{\pi t}} - e^{t}\operatorname{erfc}\sqrt{t}.$$

(iv) Finally this last part can be deduced by using the formula

$$\frac{1}{\sqrt{s}+1} - \frac{1}{\sqrt{s}-1} = -2\frac{1}{s-1}.$$

The answer is

$$\mathcal{L}^{-1}\left\{\frac{1}{\sqrt{s}-1}\right\} = \frac{1}{\sqrt{\pi t}} + e^t(1+\operatorname{erf}\sqrt{t}).$$

6. This problem is best tackled by use of power series, especially so as there is no problem manipulating them as these exponential and related functions have series that are uniformly convergent for all finite values of t and s, excluding $s = 0$. The power series for the error function (obtained by integrating the exponential series term by term) yields:

$$\operatorname{erf}\left(\frac{1}{s}\right) = \frac{2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n}{s^{2n+1}(2n+1)n!}.$$

Taking the inverse Laplace transform using linearity and standard forms gives

$$\phi(t) = \frac{2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)n!}\frac{t^{2n}}{(2n)!}.$$

After some tidying up, this implies that

$$\phi(\sqrt{t}) = \frac{2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n t^n}{(2n+1)!n!}.$$

Taking the Laplace transform of this series term by term gives

$$\mathcal{L}\left\{\phi(\sqrt{t})\right\} = \frac{2}{\sqrt{\pi s}}\sum_{n=0}^{\infty}(-1)^n\frac{(1/\sqrt{s})^{2n+1}}{(2n+1)!}$$

which is

$$\mathcal{L}\left\{\phi(\sqrt{t})\right\} = \frac{2}{\sqrt{\pi s}}\sin\left(\frac{1}{\sqrt{s}}\right)$$

as required.

7. This problem is tackled in a very similar way to the previous one. We simply integrate the series term by term and have to recognise

$$\sum_{k=0}^{\infty}\frac{(-1)^k(2k)!}{(k!)^2}\left(\frac{x}{s}\right)^{2k}$$

as the binomial series for

$$\left(1+\frac{x^2}{s^2}\right)^{-1/2}.$$

Again, the series are uniformly convergent except for $s = \pm ix$ which must be excluded alongside $s = 0$.

8. The integrand

$$\frac{\cosh(x\sqrt{s})}{s\cosh(\sqrt{s})}$$

has a singularity at the origin and wherever $\sqrt{s}$ is an odd multiple of $\pi/2$. The presence of the square roots leads one to expect branch points, but in fact there are only simple poles. There are however infinitely many of them at locations

$$s = -\left(n + \frac{1}{2}\right)^2 \pi^2$$

and at the origin. The (uncut) Bromwich contour can thus be used; all the singularities are certainly to the left of the line $s = \sigma$ in Fig. 8.9. The inverse is thus

$$\frac{1}{2\pi i}\int_{Br} e^{st}\frac{\cosh(x\sqrt{s})}{s\cosh(\sqrt{s})}ds = \text{sum of residues.}$$

The residue at $s = 0$ is straightforwardly

$$\lim_{s\to 0}(s-0)\left\{\frac{e^{st}\cosh(x\sqrt{s})}{s\cosh(\sqrt{s})}\right\} = 1.$$

The residue at the other poles is also calculated using the formula, but the calculation is messier, and the result is

$$\frac{4(-1)^n}{\pi(2n-1)}e^{-(n-1/2)^2\pi^2 t}\cos\left(n-\frac{1}{2}\right)\pi x.$$

Thus we have

$$\mathcal{L}^{-1}\left\{\frac{\cosh(x\sqrt{s})}{s\cosh(\sqrt{s})}\right\} = 1 + \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{2n-1}e^{-(n-\frac{1}{2})^2\pi^2 t}\cos\left(n-\frac{1}{2}\right)\pi x.$$

9. The Bromwich contour for the function

$$e^{-s^{\frac{1}{3}}}$$

has a cut from the branch point at the origin. We can thus use the contour depicted in Fig. 8.10. As the origin has been excluded the integrand

$$e^{st-s^{\frac{1}{3}}}$$

has no singularities in the contour, so by Cauchy's theorem the integral around C' is zero. As is usual, the two parts of the integral that are curved give zero contribution as the outer radius of Γ gets larger and larger, and inner radius of the circle γ gets smaller. This is because $\cos\theta < 0$ on the left of the imaginary axis which means that the exponent in the integrand is negative on Γ, also on γ the ds contributes a zero as the radius of the circle γ decreases. The remaining contributions are

$$\int_{AB} = -\int_{\infty}^{0} e^{-xt - x^{\frac{1}{3}} e^{i\pi/3}} dx$$

and

$$\int_{CD} = -\int_{0}^{\infty} e^{-xt - x^{\frac{1}{3}} e^{-i\pi/3}} dx.$$

These combine to give

$$\int_{AB} + \int_{CD} = -\int_{0}^{\infty} e^{-xt - \frac{1}{2}x^{\frac{1}{3}}} \sin\left(\frac{x^{\frac{1}{3}}\sqrt{3}}{2}\right) dx.$$

Substituting $x = u^3$ gives the result

$$\mathcal{L}^{-1}\{e^{-s^{\frac{1}{3}}}\} = \frac{1}{2\pi i}\int_{Br} e^{st - s^{\frac{1}{3}}} ds = \frac{3}{\pi}\int_{0}^{\infty} u^2 e^{-u^3 t - \frac{1}{2}u} \sin\left(\frac{u\sqrt{3}}{2}\right) du.$$

10. Using Laplace transforms in t solving in the usual way gives the solution

$$\bar{\phi}(x, s) = \frac{1}{s^2} e^{-x\sqrt{s^2-1}}.$$

The singularity of $\bar{\phi}(x, s)$ with the largest real part is at $s = 1$. The others are at $s = -1, 0$. Expanding $\bar{\phi}(x, s)$ about $s = 1$ gives

$$\bar{\phi}(x, s) = 1 - x\sqrt{2}(s-1)^{\frac{1}{2}} + \cdots.$$

In terms of Theorem 8.2 this means that $k = 1/2$ and $a_0 = -x\sqrt{2}$. Hence the leading term in the asymptotic expansion for $\phi(x, t)$ for large t is

$$-\frac{1}{\pi} e^t \sin\left(\frac{1}{2}\pi\right)\left(-x\sqrt{2}\frac{\Gamma(3/2)}{t^{3/2}}\right)$$

whence

$$\phi(x, t) \sim \frac{x e^t}{\sqrt{2\pi t^3}}$$

as required.

Appendix B
Table of Laplace Transforms

In this table, t is a real variable, s is a complex variable and a and b are real constants. In a few entries, the real variable x also appears.

$\mathbf{f}(\mathbf{s})(=\int_0^\infty e^{-st}\mathbf{F}(\mathbf{t})dt)$	$\mathbf{F}(\mathbf{t})$
$\frac{1}{s}$	1
$\frac{1}{s^n},\ n=1,2,\ldots$	$\frac{t^{n-1}}{(n-1)!},$
$\frac{1}{s^x},\ x>0,$	$\frac{t^{x-1}}{\Gamma(x)},$
$\frac{1}{s-a}$	e^{at}
$\frac{s}{s^2+a^2}$	$\cos(at)$
$\frac{a}{s^2+a^2}$	$\sin(at)$
$\frac{s}{s^2-a^2}$	$\cosh(at)$
$\frac{a}{s^2-a^2}$	$\sinh(at)$
$\frac{1}{(s-a)(s-b)}\quad a\neq b$	$\frac{e^{bt}-e^{at}}{b-a}$
$\frac{s}{(s-a)(s-b)}\quad a\neq b$	$\frac{be^{bt}-ae^{at}}{b-a}$
$\frac{1}{(s^2+a^2)^2}$	$\frac{\sin(at)-at\cos(at)}{2a^3}$
$\frac{s}{(s^2+a^2)^2}$	$\frac{t\sin(at)}{2a}$
$\frac{s^2}{(s^2+a^2)^2}$	$\frac{\sin(at)+at\cos(at)}{2a}$
$\frac{s^3}{(s^2+a^2)^2}$	$\cos(at)-\frac{1}{2}at\sin(at)$

P. Dyke, *An Introduction to Laplace Transforms and Fourier Series*,
Springer Undergraduate Mathematics Series, DOI: 10.1007/978-1-4471-6395-4,

$\mathbf{f(s)}(=\int_0^\infty e^{-st}\mathbf{F(t)}dt)$	$\mathbf{F(t)}$
$\dfrac{1}{\sqrt{s+a}+\sqrt{s+b}}$	$\dfrac{e^{-bt}-e^{-at}}{2(b-a)\sqrt{\pi t^3}}$
$\dfrac{e^{-a/s}}{\sqrt{s}}$	$\dfrac{\cos 2\sqrt{at}}{\sqrt{\pi t}}$
$\dfrac{e^{-a/s}}{s\sqrt{s}}$	$\dfrac{\sin 2\sqrt{at}}{\sqrt{\pi a}}$
$\dfrac{e^{-a/s}}{s^{n+1}}$	$\left(\dfrac{t}{a}\right)^{n/2} J_n(2\sqrt{at})$
1	$\delta(t)$
s^n	$\delta^{(n)}(t)$
$\dfrac{e^{-as}}{s}$	$H(t-a)$
$\dfrac{e^{-a\sqrt{s}}}{s}$	$\operatorname{erfc}\left(\dfrac{a}{2\sqrt{t}}\right)$
$e^{-a\sqrt{s}}$	$\dfrac{a}{2\sqrt{\pi t^3}}e^{-a^2/4t}$
$\dfrac{1}{s\sqrt{s+a}}$	$\dfrac{\operatorname{erf}\sqrt{at}}{\sqrt{a}}$
$\dfrac{1}{\sqrt{s}(s-a)}$	$\dfrac{e^{at}\operatorname{erf}\sqrt{at}}{\sqrt{a}}$
$\dfrac{1}{\sqrt{s-a}+b}$	$e^{at}\left\{\dfrac{1}{\sqrt{\pi t}} - be^{b^2 t}\operatorname{erfc}(b\sqrt{t})\right\}$
$\dfrac{1}{\sqrt{s^2+a^2}}$	$J_0(at)$
$\tan^{-1}(a/s)$	$\dfrac{\sin(at)}{t}$
$\dfrac{\sinh(sx)}{s\sinh(sa)}$	$\dfrac{x}{a}+\dfrac{2}{\pi}\sum_{n=1}^{\infty}\dfrac{(-1)^n}{n}\sin\left(\dfrac{n\pi x}{a}\right)\cos\left(\dfrac{n\pi t}{a}\right)$
$\dfrac{\sinh(sx)}{s\cosh(sa)}$	$\dfrac{4}{\pi}\sum_{n=1}^{\infty}\dfrac{(-1)^n}{2n-1}\sin\left(\dfrac{(2n-1)\pi x}{2a}\right)\sin\left(\dfrac{(2n-1)\pi t}{2a}\right)$
$\dfrac{\cosh(sx)}{s\cosh(sa)}$	$1+\dfrac{4}{\pi}\sum_{n=1}^{\infty}\dfrac{(-1)^n}{2n-1}\cos\left(\dfrac{(2n-1)\pi x}{2a}\right)\cos\left(\dfrac{(2n-1)\pi t}{2a}\right)$
$\dfrac{\cosh(sx)}{s\sinh(sa)}$	$\dfrac{t}{a}+\dfrac{2}{\pi}\sum_{n=1}^{\infty}\dfrac{(-1)^n}{n}\cos\left(\dfrac{n\pi x}{a}\right)\sin\left(\dfrac{n\pi t}{a}\right)$
$\dfrac{\sinh(sx)}{s^2\cosh(sa)}$	$x+\dfrac{8a}{\pi^2}\sum_{n=1}^{\infty}\dfrac{(-1)^n}{(2n-1)^2}\cos\left(\dfrac{(2n-1)\pi x}{2a}\right)\sin\left(\dfrac{(2n-1)\pi t}{2a}\right)$
$\dfrac{\sinh(x\sqrt{s})}{s\sinh(a\sqrt{s})}$	$\dfrac{x}{a}+\dfrac{2}{\pi}\sum_{n=1}^{\infty}\dfrac{(-1)^n}{n}e^{-n^2\pi^2 t/a^2}\sin\left(\dfrac{n\pi x}{a}\right)$
$\dfrac{\cosh(x\sqrt{s})}{s\cosh(a\sqrt{s})}$	$1+\dfrac{4}{\pi}\sum_{n=1}^{\infty}\dfrac{(-1)^n}{2n-1}e^{-(2n-1)^2\pi^2 t/4a^2}\cos\left(\dfrac{(2n-1)\pi x}{2a}\right)$
$\dfrac{\sinh(x\sqrt{s})}{s^2\sinh(a\sqrt{s})}$	$\dfrac{xt}{a}+\dfrac{2a^2}{\pi^3}\sum_{n=1}^{\infty}\dfrac{(-1)^n}{n^3}(1-e^{-n^2\pi^2 t/a^2})\sin\left(\dfrac{n\pi x}{a}\right)$

In the last four entries $\gamma = 0.5772156\ldots$ is the Euler-Mascheroni constant. The next four Laplace transforms are of periodic functions that are given diagrammatically. The two column format is abandoned.

$\mathbf{f(s)}(=\int_0^\infty e^{-st}\mathbf{F(t)}dt)$	$\mathbf{F(t)}$
$\frac{1}{as^2}\tanh\left(\frac{as}{2}\right)$	$F(t)=\begin{cases} t/a & 0\le t\le a \\ 2-t/a & a<t\le 2a \end{cases}$ $F(t)=F(t+2a)$
$\frac{1}{s}\tanh\left(\frac{as}{2}\right)$	$F(t)=\begin{cases} 1 & 0\le t\le a \\ -1 & a<t\le 2a \end{cases}$ $F(t)=F(t+2a)$
$\frac{e^{-a/s}}{s^{n+1}}$	$\left(\frac{t}{a}\right)^{n/2} J_n(2\sqrt{at})$
$\frac{\pi a}{a^2+s^2}\coth\left(\frac{as}{2}\right)$	$\left\lvert\sin\left(\frac{\pi t}{a}\right)\right\rvert$
$\frac{\pi a}{(a^2s^2+\pi^2)(1-e^{-as})}$	$F(t)=\left\{\begin{matrix} \sin(\frac{\pi t}{a}) & 0\le t\le a \\ 0 & a<t\le 2a \end{matrix}\right\}$ $F(t)=F(t+2a)$
$\frac{1}{as^2}-\frac{e^{-as}}{s(1-e^{-as})}$	$F(t)=t/a,\quad 0\le t\le a\quad F(t)=F(t+a)$
$\ln\left(\frac{s+a}{s+b}\right)$	$\frac{e^{-bt}-e^{-at}}{t}$
$\ln\left(\frac{s^2+a^2}{s^2+b^2}\right)$	$\frac{2(\cos bt-\cos at)}{t}$
$\frac{1}{s^3+a^3}$	$\frac{e^{at/2}}{3a^2}\left\{\sqrt{3}\sin\frac{\sqrt{3}at}{2}-\cos\frac{\sqrt{3}at}{2}+e^{-3at/2}\right\}$
$\frac{s}{s^3+a^3}$	$\frac{e^{at/2}}{3a^2}\left\{\sqrt{3}\sin\frac{\sqrt{3}at}{2}+\cos\frac{\sqrt{3}at}{2}-e^{-3at/2}\right\}$
$\frac{s^2}{s^3+a^3}$	$\frac{1}{3}\left(e^{-at}+2e^{at/2}\cos\frac{\sqrt{3}at}{2}\right)$
$\frac{1}{s^3-a^3}$	$\frac{e^{-at/2}}{3a^2}\left\{e^{3at/2}-\sqrt{3}\sin\frac{\sqrt{3}at}{2}-\cos\frac{\sqrt{3}at}{2}\right\}$
$\frac{s}{s^3-a^3}$	$\frac{e^{-at/2}}{3a^2}\left\{\sqrt{3}\sin\frac{\sqrt{3}at}{2}-\cos\frac{\sqrt{3}at}{2}+e^{3at/2}\right\}$
$\frac{s^2}{s^3-a^3}$	$\frac{1}{3}\left(e^{at}+2e^{-at/2}\cos\frac{\sqrt{3}at}{2}\right)$
$\frac{1}{s^4+4a^4}$	$\frac{1}{4a^3}(\sin at\cosh at-\cos at\sinh at)$
$\frac{s}{s^4+4a^4}$	$\frac{\sin at\sinh at}{2a^2}$
$-\frac{\gamma+\ln s}{s}$	$\ln t$
$\frac{\pi^2}{6s}+\frac{(\gamma+\ln s)^2}{s}$	$\ln^2 t$
$\frac{\ln s}{s}$	$-(\ln t+\gamma)$
$\frac{\ln^2 s}{s}$	$-(\ln t+\gamma)^2-\frac{1}{6}\pi^2$

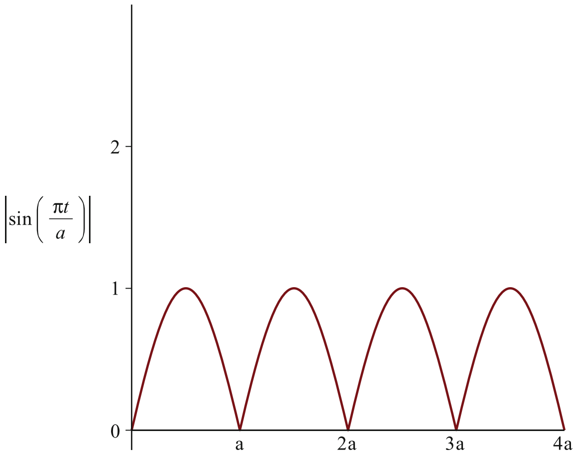


Fig. B.1 The Laplace transform of the above function, the rectified sine wave $F(t) = |\sin\left(\frac{\pi t}{a}\right)|$ is given by $f(s) = \frac{\pi a}{a^2 s^2 + \pi^2} \coth\left(\frac{as}{2}\right)$

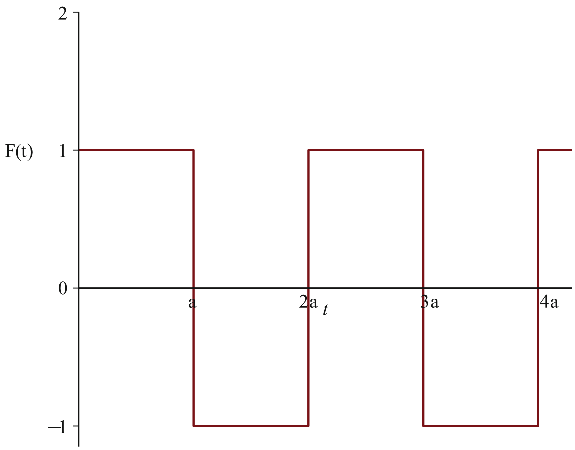


Fig. B.2 The Laplace transform of the above square wave function is given by $f(s) = \frac{1}{s} \tanh\left(\frac{1}{2}as\right)$

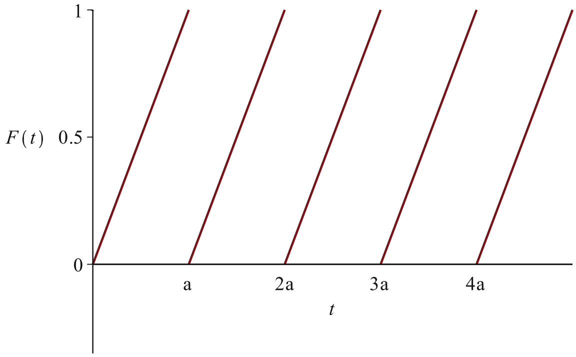


Fig. B.3 The Laplace transform of the above saw-tooth function is given by $f(s) = \frac{1}{as^2} - \frac{e^{-as}}{s(1-e^{-as})}$

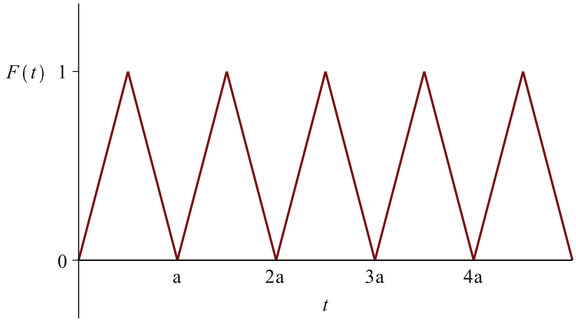


Fig. B.4 The Laplace transform of the above saw-tooth function is given by $f(s) = \frac{1}{as^2}\tanh\left(\frac{1}{2}as\right)$

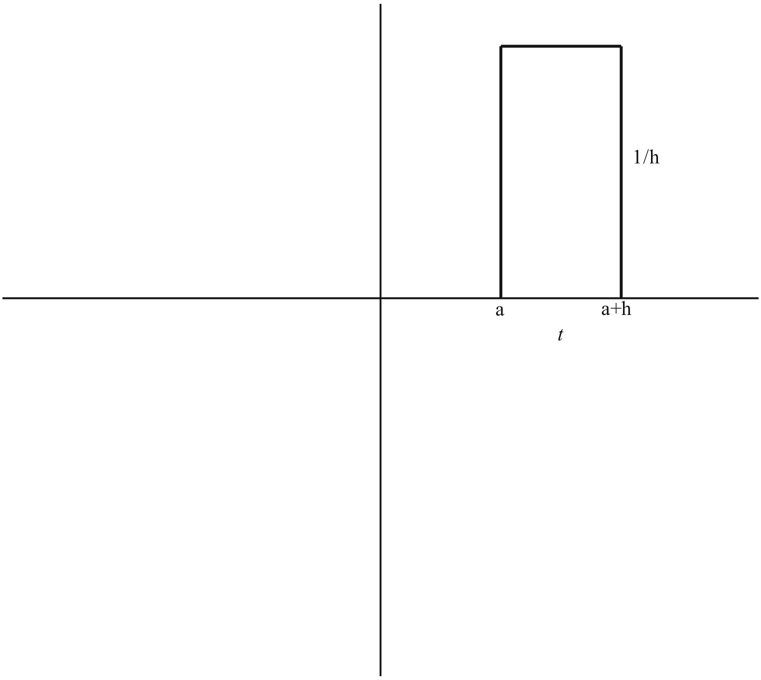


Fig. B.5 The Laplace transform of the above top hat function is given by $f(s) = \frac{e^{-as}}{sh}(1 - e^{-sh})$. Thus the Haar wavelet that corresponds to $a = 0$, $h = 1$ has the Laplace transform $\frac{1 - e^{-s}}{s}$

Appendix C
Linear Spaces

C.1 Linear Algebra

In this appendix, some fundamental concepts of linear algebra are given. The proofs are largely omitted; students are directed to textbooks on linear algebra for these. For this subject, we need to be precise in terms of the basic mathematical notions and notations we use. Therefore we uncharacteristically employ a formal mathematical style of prose. It is essential to be rigorous with the basic mathematics, but it is often the case that an over formal treatment can obscure rather than enlighten. That is why this material appears in an appendix rather than in the main body of the text.

A set of objects (called the elements of the set) is written as a sequence of (usually) lower case letters in between braces:-

$$A = \{a_1, a_2, a_3, \ldots, a_n\}.$$

In discussing Fourier series, sets have infinitely many elements, so there is a row of dots after a_n too. The symbol $\in$ read as "belongs to" should be familiar to most. So $s \in S$ means that s is a member of the set S. Sometimes the alternative notation

$$S = \{x | f(x)\}$$

is used. The vertical line is read as "such that" so that $f(x)$ describes some property that x possess in order that $s \in S$. An example might be

$$S = \{x | x \in \mathbb{R}, |x| \leq 2\}$$

so S is the set of real numbers that lie between -2 and $+2$.

The following notation is standard but is reiterated here for reference:
(a, b) denotes the open interval $\{x|a < x < b\}$,
$[a, b]$ denotes the closed interval $\{x|a \leq x \leq b\}$,
$[a, b)$ is the set $\{x|a \leq x < b\}$,
and $(a, b]$ is the set $\{x|a < x \leq b\}$.

P. Dyke, *An Introduction to Laplace Transforms and Fourier Series*,
Springer Undergraduate Mathematics Series, DOI: 10.1007/978-1-4471-6395-4,

The last two are described as half closed or half open intervals and are reasonably obvious extensions of the first two definitions. In Fourier series, ∞ is often involved so the following intervals occur:-

$$(a, \infty) = \{x|a < x\} \quad [a, \infty) = \{x|a \leq x\}$$
$$(-\infty, a) = \{x|x < a\} \quad (-\infty, a] = \{x|x \leq a\}.$$

These are all obvious extensions. Where appropriate, use is also made of the following standard sets

$$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}, \text{ the set of integers}$$
$$\mathbb{Z}_+ \text{ is the set of positive integers including zero: } \{0, 1, 2, 3, \ldots\}$$
$$\mathbb{R} = \{x|x \text{ is a real number}\} = (-\infty, \infty).$$

Sometimes (but very rarely) we might use the set of fractions or rationals $\mathbb{Q}$:

$$\mathbb{Q} = \left\{\frac{m}{n} \mid m \text{ and } n \text{ are integers }, n \neq 0\right\}.$$

$\mathbb{R}_+$ is the set of positive real numbers

$$\mathbb{R}_+ = \{x \mid x \in \mathbb{R}, x \geq 0\}.$$

Finally the set of complex numbers $\mathbb{C}$ is defined by

$$\mathbb{C} = \{z = x + iy \mid x, y \in \mathbb{R}, i = \sqrt{-1}\}.$$

The standard notation

$$x = \Re\{z\}, \quad \text{the real part of } z$$
$$y = \Im\{z\}, \quad \text{the imaginary part of } z$$

has already been met in Chap. 1.

Hopefully, all of this is familiar to most of you. We will need these to define the particular normed spaces within which Fourier series operate. This we now proceed to do. A vector space V is an algebraic structure that consists of elements (called vectors) and two operations (called addition $+$ and multiplication $\times$). The following gives a list of properties obeyed by vectors **a,b,c** and scalars $\alpha, \beta \in F$ where F is a field (usually $\mathbb{R}$ or $\mathbb{C}$).

1. $\mathbf{a} + \mathbf{b}$ is also a vector (closure under addition).
2. $(\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c})$ (associativity under addition).
3. There exists a zero vector denoted by $\mathbf{0}$ such that $\mathbf{0} + \mathbf{a} = \mathbf{a} + \mathbf{0} = \mathbf{a} \;\; \forall \mathbf{a} \in V$ (additive identity).

4. For every vector $\mathbf{a} \in V$ there is a vector $-\mathbf{a}$ (called "minus $\mathbf{a}$" such that $\mathbf{a} + (-\mathbf{a}) = \mathbf{0}$.
5. $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$ for every $\mathbf{a}, \mathbf{b} \in V$ (additive commutativity).
6. $\alpha\mathbf{a} \in V$ for every $\alpha \in F, \mathbf{a} \in V$ (scalar multiplicity).
7. $\alpha(\mathbf{a} + \mathbf{b}) = \alpha\mathbf{a} + \alpha\mathbf{b}$ for every $\alpha \in F, \mathbf{a}, \mathbf{b} \in V$ (first distributive law).
8. $(\alpha + \beta)\mathbf{a} = \alpha\mathbf{a} + \beta\mathbf{a}$ for every $\alpha, \beta \in F, \mathbf{a} \in V$ (second distributive law).
9. For the unit scalar 1 of the field F, and every $\mathbf{a} \in V$ $1 \cdot \mathbf{a} = \mathbf{a}$ (multiplicative identity).

The set V whose elements obey the above nine properties over a field F is called a vector space over F. The name linear space is also used in place of vector space and is useful as the name "vector" conjures up mechanics to many and gives a false impression in the present context. In the study of Fourier series, vectors are in fact functions. The name linear space emphasises the linearity property which is confirmed by the following definition and properties.

Definition C.1 *If* $\mathbf{a_1}, \mathbf{a_2}, \ldots, \mathbf{a_n} \in V$ *where* V *is a linear space over a field* F *and if there exist scalars* $\alpha_1, \alpha_2, \ldots, \alpha_n \in F$ *such that*

$$\mathbf{b} = \alpha_1\mathbf{a_1} + \alpha_2\mathbf{a_2} + \cdots + \alpha_n\mathbf{a_n}$$

(called a linear combination *of the vectors* $\mathbf{a_1}, \mathbf{a_2}, \ldots, \mathbf{a_n}$*) then the collection of all such* $\mathbf{b}$ *which are a linear combination of the vectors* $\mathbf{a_1}, \mathbf{a_2}, \ldots, \mathbf{a_n}$ *is called the* span *of* $\mathbf{a_1}, \mathbf{a_2}, \ldots, \mathbf{a_n}$ *denoted by span* $\{\mathbf{a_1}, \mathbf{a_2}, \ldots, \mathbf{a_n}\}$.

If this definition seems innocent, then the following one which depends on it is not. It is one of the most crucial properties possibly in the whole of mathematics.

Definition C.2 (linear independence) *If* V *is a linear (vector) space, the vectors* $\mathbf{a_1}, \mathbf{a_2}, \ldots, \mathbf{a_n} \in V$ *are said to be linearly independent if the equation*

$$\alpha_1\mathbf{a_1} + \alpha_2\mathbf{a_2} + \cdots + \alpha_n\mathbf{a_n} = \mathbf{0}$$

implies that all of the scalars are zero, i.e.

$$\alpha_1 = \alpha_2 = \ldots = \alpha_n = 0, (\alpha_1, \alpha_2, \ldots, \alpha_n \in F).$$

Otherwise, $\alpha_1, \alpha_2, \ldots, \alpha_n$ *are said to be linearly dependent.*

Again, it is hoped that this is not a new concept. However, here is an example. Most texts take examples from geometry, true vectors indeed. This is not appropriate here so instead this example is algebraic.

Example C.1 *Is the set* $S = \{1, x, 2 + x, x^2\}$ *with* $F = \mathbf{R}$ *linearly independent?*

Solution The most general combination of $1, x, 2 + x, x^2$ is

$$y = \alpha_1 + \alpha_2 x + \alpha_3(2 + x) + \alpha_4 x^2$$

where x is a variable that can take any real value.

Now, $y = 0$ for all x does not imply $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$, for if we choose $\alpha_1 + 2\alpha_2 = 0, \alpha_2 + \alpha_3 = 0$ and $\alpha_4 = 0$ then $y = 0$. The combination $\alpha_1 = 1, \alpha_3 = -\frac{1}{2}, \alpha_2 = \frac{1}{2}, \alpha_4 = 0$ will do. The set is therefore not linearly independent.

On the other hand, the set $\{1, x, x^2\}$ is most definitely linearly independent as

$$\alpha_1 + \alpha_2 x + \alpha_3 x^2 = 0 \text{ for all } x \Rightarrow \alpha_1 = \alpha_2 = \alpha_3 = 0.$$

It is possible to find many independent sets. One could choose $\{x, \sin x, \ln x\}$ for example: however sets like this are not very useful as they do not lead to any applications. The set $\{1, x, x^2\}$ spans all quadratic functions. Here is another definition that we hope is familiar.

Definition C.3 *A finite set of vectors* $\mathbf{a_1}, \mathbf{a_2}, \ldots, \mathbf{a_n}$ *is said to be a* basis *for the linear space* V *if the set of vectors* $\mathbf{a_1}, \mathbf{a_2}, \ldots, \mathbf{a_n}$ *is linearly independent and* $V = span\{\mathbf{a_1}, \mathbf{a_2}, \ldots, \mathbf{a_n}\}$ *The natural number* n *is called the dimension of* V *and we write* $n = dim(V)$.

Example C.2 *Let* $[a, b]$ *(with* $a < b$*) denote the finite closed interval as already defined. Let* f *be a continuous real valued function whose value at the point* x *of* $[a, b]$ *is* $f(x)$. *Let* $C[a, b]$ *denote the set of all such functions. Now, if we define addition and scalar multiplication in the natural way, i.e.* $f_1 + f_2$ *is simply the value of* $f_1(x) + f_2(x)$ *and similarly* αf *is the value of* $\alpha f(x)$, *then it is clear that* $C[a, b]$ *is a real vector space. In this case, it is clear that the set* $x, x^2, x^3, \ldots, x^n$ *are all members of* $C[a, b]$ *for arbitrarily large* n. *It is therefore not possible for* $C[a, b]$ *to be finite dimensional.*

Perhaps it is now a little clearer as to why the set $\{1, x, x^2\}$ is useful as this is a basis for all quadratics, whereas $\{x, \sin x, \ln x\}$ does not form a basis for any well known space. Of course, there is usually an infinite choice of basis for any particular linear space. For the quadratic functions the sets $\{1-x, 1+x, x^2\}$ or $\{1, 1-x^2, 1+2x+x^2\}$ will do just as well. That we have these choices of bases is useful and will be exploited later.

Most books on elementary linear algebra are content to stop at this point and consolidate the above definitions through examples and exercises. However, we need a few more definitions and properties in order to meet the requirements of a Fourier series.

Definition C.4 *Let* V *be a real or complex linear space. (That is the field* F *over which the space is defined is either* $\mathbb{R}$ *or* $\mathbb{C}$.*) An inner product is an operation between two elements of* V *which results in a scalar. This scalar is denoted by* $\langle \mathbf{a_1}, \mathbf{a_2} \rangle$ *and has the following properties:-*

1. *For each* $\mathbf{a_1} \in V$, $\langle \mathbf{a_1}, \mathbf{a_1} \rangle$ *is a non-negative real number, i.e.*

$$\langle \mathbf{a_1}, \mathbf{a_1} \rangle \geq 0.$$

2. *For each* $\mathbf{a_1} \in V$, $\langle \mathbf{a_1}, \mathbf{a_1} \rangle = 0$ *if and only if* $\mathbf{a_1} = \mathbf{0}$.
3. *For each* $\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3} \in V$ *and* $\alpha_1, \alpha_2 \in F$

$$\langle \alpha_1 \mathbf{a}_1 + \alpha_2 \mathbf{a}_2, \mathbf{a}_3 \rangle = \alpha_1 \langle \mathbf{a}_1, \mathbf{a}_3 \rangle + \alpha_2 \langle \mathbf{a}_2, \mathbf{a}_3 \rangle.$$

4. *For each* $\mathbf{a_1}, \mathbf{a_2} \in V$, $\langle \mathbf{a_1}, \mathbf{a_2} \rangle = \overline{\langle \mathbf{a_2}, \mathbf{a_1} \rangle}$

where the overbar in the last property denotes the complex conjugate. If $F = \mathbb{R}$ α_1, α_2 *are real, and Property 4 becomes obvious.*

No doubt, students who are familiar with the geometry of vectors will be able to identify the inner product $\langle \mathbf{a_1}, \mathbf{a_2} \rangle$ with $\mathbf{a}_1.\mathbf{a}_2$ the scalar product of the two vectors $\mathbf{a}_1$ and $\mathbf{a}_2$. This is one useful example, but it is by no means essential to the present text where most of the inner products take the form of integrals.

Inner products provide a rich source of properties that would be out of place to dwell on or prove here. For example:

$$\langle \mathbf{0}, \mathbf{a} \rangle = 0 \quad \forall\, \mathbf{a} \in V$$

and

$$\langle \alpha \mathbf{a}_1, \alpha \mathbf{a}_2 \rangle = |\alpha|^2 \langle \mathbf{a}_1, \mathbf{a}_2 \rangle.$$

Instead, we introduce two examples of inner product spaces.

1. If $\mathbb{C}^n$ is the vector space V, i.e. a typical element of V has the form $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ where $a_r = x_r + i y_r$, x_r, $y_r \in \mathbb{R}$. The inner product $\langle \mathbf{a}, \mathbf{b} \rangle$ is defined by
$$\langle \mathbf{a}, \mathbf{b} \rangle = a_1 \overline{b_1} + a_2 \overline{b_2} + \cdots + a_n \overline{b_n},$$
the overbar denoting complex conjugate.
2. Nearer to our applications of inner products is the choice $V = C[a, b]$ the linear space of all continuous functions f defined on the closed interval $[a, b]$. With the usual summation of functions and multiplication by scalars this can be verified to be a vector space over the field of complex numbers $\mathbb{C}^n$. Given a pair of continuous functions f, g we can define their inner product by
$$\langle f, g \rangle = \int_a^b f(x) \overline{g(x)} dx.$$
It is left to the reader to verify that this is indeed an inner product space satisfying the correct properties in Definition C.4.

It is quite typical for linear spaces involving functions to be infinite dimensional. In fact it is very unusual for it to be otherwise.

What has been done so far is to define a linear space and an inner product on that space. It is nearly always true that we can define what is called a “norm” on a linear space. The norm is independent of the inner product in theory, but there is almost always a connection in practice. The norm is a generalisation of the notion of distance. If the linear space is simply two or three dimensional vectors, then the norm can indeed be distance. It is however, even in this case possible to define others. Here is the general definition of norm.

Definition C.5 *Let V be a linear space. A norm on V is a function from V to $\mathbb{R}_+$ (non-negative real numbers), denoted by being placed between two vertical lines $||\cdot||$ which satisfies the following four criteria:-*

1. *For each $\mathbf{a}_1 \in V$, $||\mathbf{a}_1|| \geq 0$.*
2. *$||\mathbf{a}_1|| = 0$ if and only if $\mathbf{a}_1 = \mathbf{0}$.*
3. *For each $\mathbf{a}_1 \in V$ and $\alpha \in \mathbb{C}$*

$$||\alpha\mathbf{a}_1|| = |\alpha|||\mathbf{a}_1||.$$

4. *For every $\mathbf{a}_1, \mathbf{a}_2 \in V$*

$$||\mathbf{a}_1 + \mathbf{a}_2|| \leq ||\mathbf{a}_1|| + ||\mathbf{a}_2||.$$

(4 is the triangle inequality.)

For the vector space comprising the elements $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ where $a_r = x_r + iy_r$, $x_r, y_r \in \mathbb{R}$, i.e. $\mathbb{C}^n$ met previously, the obvious norm is

$$\begin{aligned} ||\mathbf{a}|| &= [|a_1|^2 + |a_2|^2 + |a_3|^2 + \cdots + |a_n|^2]^{1/2} \\ &= [\langle\mathbf{a}, \mathbf{a}\rangle]^{1/2}. \end{aligned}$$

It is true in general that we can always define the norm $||\cdot||$ of a linear space equipped with an inner product $\langle ., . \rangle$ to be such that

$$||\mathbf{a}|| = [\langle\mathbf{a}, \mathbf{a}\rangle]^{1/2}.$$

This norm is used in the next example. A linear space equipped with an inner product is called an inner product space. The norm induced by the inner product, sometimes called the *natural* norm for the function space $C[a, b]$, is

$$||f|| = \left[\int_a^b |f|^2 dx\right]^{1/2}.$$

For applications to Fourier series we are able to make $||f|| = 1$ and we adjust elements of V, i.e. $C[a, b]$ so that this is achieved. This process is called normalisation. Linear spaces with special norms and other properties are the directions in which this subject now naturally moves. The interested reader is directed towards books on functional analysis.

We now establish an important inequality called the Cauchy–Schwarz inequality. We state it in the form of a theorem and prove it.

Theorem C.1 (Cauchy–Schwarz) *Let V be a linear space with inner product $\langle ., . \rangle$, then for each $\mathbf{a}, \mathbf{b} \in V$ we have:*

$$|\langle \mathbf{a}, \mathbf{b} \rangle|^2 \leq ||\mathbf{a}|| \cdot ||\mathbf{b}||.$$

Proof If $\langle \mathbf{a}, \mathbf{b} \rangle = 0$ then the result is self evident. We therefore assume that $\langle \mathbf{a}, \mathbf{b} \rangle = \alpha \neq 0$, α may of course be complex. We start with the inequality

$$||\mathbf{a} - \lambda\alpha\mathbf{b}||^2 \geq 0$$

where λ is a real number. Now,

$$||\mathbf{a} - \lambda\alpha\mathbf{b}||^2 = \langle \mathbf{a} - \lambda\alpha\mathbf{b}, \mathbf{a} - \lambda\alpha\mathbf{b} \rangle.$$

We use the properties of the inner product to expand the right hand side as follows:-

$$\begin{aligned}
\langle \mathbf{a} - \lambda\alpha\mathbf{b}, \mathbf{a} - \lambda\alpha\mathbf{b} \rangle &= \langle \mathbf{a}, \mathbf{a} \rangle - \lambda\langle \alpha\mathbf{b}, \mathbf{a} \rangle - \lambda\langle \mathbf{a}, \alpha\mathbf{b} \rangle + \lambda^2|\alpha|^2\langle \mathbf{b}, \mathbf{b} \rangle \geq 0 \\
&\text{so } ||\mathbf{a}||^2 - \lambda\alpha\langle \mathbf{b}, \mathbf{a} \rangle - \lambda\bar{\alpha}\langle \mathbf{a}, \mathbf{b} \rangle + \lambda^2|\alpha|^2||\mathbf{b}||^2 \geq 0 \\
&\text{i.e. } ||\mathbf{a}||^2 - \lambda\alpha\bar{\alpha} - \lambda\bar{\alpha}\alpha + \lambda^2|\alpha|^2||\mathbf{b}||^2 \geq 0 \\
&\text{so } ||\mathbf{a}||^2 - 2\lambda|\alpha|^2 + \lambda^2|\alpha|^2||\mathbf{b}||^2 \geq 0.
\end{aligned}$$

This last expression is a quadratic in the real parameter λ, and it has to be positive for all values of λ. The condition for the quadratic

$$a\lambda^2 + b\lambda + c$$

to be non-negative is that $b^2 \leq 4ac$ and $a > 0$. With

$$a = |\alpha|^2||\mathbf{b}||^2, \quad b = -2|\alpha|^2, \quad c = ||\mathbf{a}||^2$$

the inequality $b^2 \leq 4ac$ is

$$\begin{aligned}
4|\alpha|^4 &\leq 4|\alpha|^2||\mathbf{a}||^2||\mathbf{b}||^2 \\
\text{or } |\alpha|^2 &\leq ||\mathbf{a}||||\mathbf{b}||
\end{aligned}$$

and since $\alpha = \langle \mathbf{a}, \mathbf{b} \rangle$ the result follows. □

The following is an example that typifies the process of proving that something is a norm.

Example C.3 *Prove that $||\mathbf{a}|| = \sqrt{\langle \mathbf{a}.\mathbf{a} \rangle} \in V$ is indeed a norm for the vector space V with inner product $\langle , \rangle$.*

Proof The proof comprises showing that $\sqrt{\langle \mathbf{a}, \mathbf{a}\rangle}$ satisfies the four properties of a norm.

1. $||\mathbf{a}|| \geq 0$ follows immediately from the definition of square roots.
2. If $\mathbf{a} = \mathbf{0} \iff \sqrt{\langle \mathbf{a}, \mathbf{a}\rangle} = 0$.
3.

$$\begin{aligned} ||\alpha\mathbf{a}|| &= \sqrt{\langle \alpha\mathbf{a}, \alpha\mathbf{a}\rangle} = \\ \sqrt{\alpha\bar{\alpha}\langle \mathbf{a}, \mathbf{a}\rangle} &= \sqrt{|\alpha|^2\langle \mathbf{a}, \mathbf{a}\rangle} = \\ |\alpha|\sqrt{\langle \mathbf{a}, \mathbf{a}\rangle} &= |\alpha|||\mathbf{a}||. \end{aligned}$$

4. This fourth property is the only one that takes a little effort to prove. Consider $||\mathbf{a}+\mathbf{b}||^2$. This is equal to

$$\begin{aligned} \langle \mathbf{a}+\mathbf{b}, \mathbf{a}+\mathbf{b}\rangle &= \langle \mathbf{a}, \mathbf{a}\rangle + \langle \mathbf{b}, \mathbf{a}\rangle + \langle \mathbf{a}, \mathbf{b}\rangle + \langle \mathbf{b}, \mathbf{b}\rangle \\ &= ||\mathbf{a}||^2 + \langle \mathbf{a}, \mathbf{b}\rangle + \overline{\langle \mathbf{a}, \mathbf{b}\rangle} + ||\mathbf{b}||^2. \end{aligned}$$

The expression $\langle \mathbf{a}, \mathbf{b}\rangle + \overline{\langle \mathbf{a}, \mathbf{b}\rangle}$, being the sum of a number and its complex conjugate, is real. In fact

$$\begin{aligned} |\langle \mathbf{a}, \mathbf{b}\rangle + \overline{\langle \mathbf{a}, \mathbf{b}\rangle}| &= |2\Re\langle \mathbf{a}, \mathbf{b}\rangle| \\ &\leq 2|\langle \mathbf{a}, \mathbf{b}\rangle| \\ &\leq 2||\mathbf{a}|| \cdot ||\mathbf{b}|| \end{aligned}$$

using the Cauchy–Schwarz inequality. Thus

$$\begin{aligned} ||\mathbf{a}+\mathbf{b}||^2 &= ||\mathbf{a}||^2 + \langle \mathbf{a}, \mathbf{b}\rangle + \overline{\langle \mathbf{a}, \mathbf{b}\rangle} + ||\mathbf{b}||^2 \\ &\leq ||\mathbf{a}||^2 + 2||\mathbf{a}||||\mathbf{b}|| + ||\mathbf{b}||^2 \\ &= (||\mathbf{a}+\mathbf{b}||)^2. \end{aligned}$$

Hence $||\mathbf{a}+\mathbf{b}|| \leq ||\mathbf{a}||+||\mathbf{b}||$ which establishes the triangle inequality, Property 4.

Hence $||\mathbf{a}|| = \sqrt{\langle \mathbf{a}, \mathbf{a}\rangle}$ is a norm for V. □

An important property associated with linear spaces is orthogonality. It is a direct analogy/generalisation of the geometric result $\mathbf{a}.\mathbf{b} = 0, \mathbf{a} \neq \mathbf{0}, \mathbf{b} \neq \mathbf{0}$ if $\mathbf{a}$ and $\mathbf{b}$ represent directions that are at right angles (i.e. are orthogonal) to each other. This idea leads to the following pair of definitions.

Definition C.6 *Let V be an inner product space, and let $\mathbf{a}, \mathbf{b} \in V$. If $\langle \mathbf{a}, \mathbf{b}\rangle = 0$ then vectors $\mathbf{a}$ and $\mathbf{b}$ are said to be orthogonal.*

Definition C.7 *Let V be a linear space, and let $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$ be a sequence of vectors, $\mathbf{a}_r \in V$, $a_r \neq 0$, $r = 1, 2, \ldots, n$, and let $\langle \mathbf{a}_i, \mathbf{a}_j \rangle = 0$, $i \neq j$, $0 \leq i, j \leq n$. Then $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$ is called an orthogonal set of vectors.*

Further to these definitions, a vector $\mathbf{a} \in V$ for which $||\mathbf{a}|| = 1$ is called a unit vector and if an orthogonal set of vectors consists of all unit vectors, the set is called *orthonormal*.

It is also possible to let $n \to \infty$ and obtain an orthogonal set for an infinite dimensional inner product space. We make use of this later in this chapter, but for now let us look at an example.

Example C.4 *Determine an orthonormal set of vectors for the linear space that consists of all real linear functions:*

$$\{a + bx : a, b \in \mathbb{R} \;\; 0 \leq x \leq 1\}$$

using as inner product

$$\langle f, g \rangle = \int_0^1 fg dx.$$

Solution The set $\{1, x\}$ forms a basis, but it is not orthogonal. Let $a + bx$ and $c + dx$ be two vectors. In order to be orthogonal we must have

$$\langle a + bx, c + dx \rangle = \int_0^1 (a + bx)(c + dx) dx = 0.$$

Performing the elementary integration gives the following condition on the constants a, b, c, and d

$$ac + \frac{1}{2}(bc + ad) + \frac{1}{3}bd = 0.$$

In order to be orthonormal too we also need

$$||a + bx|| = 1 \text{ and } ||c + dx|| = 1$$

and these give, additionally,

$$a^2 + b^2 = 1, c^2 + d^2 = 1.$$

There are four unknowns and three equations here, so we can make a convenient choice. Let us set

$$a = -b = \frac{1}{\sqrt{2}}$$

which gives

$$\frac{1}{\sqrt{2}}(1 - x)$$

as one vector. The first equation now gives $3c = -d$ from which

$$c = \frac{1}{\sqrt{10}}, \quad d = -\frac{3}{\sqrt{10}}.$$

Hence the set $\{(1-x)/\sqrt{10}, (1-3x)/\sqrt{10}\}$ is a possible orthonormal one.

Of course there are infinitely many possible orthonormal sets, the above was one simple choice. The next definition follows naturally.

Definition C.8 *In an inner product space, an orthonormal set that is also a basis is called an orthonormal basis.*

This next example involves trigonometry which at last gets us close to discussing Fourier series.

Example C.5 *Show that* $\{\sin(x), \cos(x)\}$ *is an orthogonal basis for the inner product space* $V = \{a\sin(x) + b\cos(x) \mid a, b \in \mathbb{R},\ 0 \le x \le \pi\}$ *using as inner product*

$$\langle f, g\rangle = \int_0^1 fg\,dx,\ f, g \in V$$

and determine an orthonormal basis.

Solution V is two dimensional and the set $\{\sin(x), \cos(x)\}$ is obviously a basis. We merely need to check orthogonality. First of all,

$$\begin{aligned}\langle \sin(x), \cos(x)\rangle = \int_0^\pi \sin(x)\cos(x)dx &= \frac{1}{2}\int_0^\pi \sin(2x)dx \\ &= \left[-\frac{1}{4}\cos(2x)\right]_0^\pi \\ &= 0.\end{aligned}$$

Hence orthogonality is established. Also,

$$\langle \sin(x), \sin(x)\rangle = \int_0^\pi \sin^2(x)dx = \frac{\pi}{2}$$

and

$$\langle \cos(x), \cos(x)\rangle = \int_0^\pi \cos^2(x)dx = \frac{\pi}{2}.$$

Therefore

$$\left\{\sqrt{\frac{2}{\pi}}\sin(x), \sqrt{\frac{2}{\pi}}\cos(x)\right\}$$

is an orthonormal basis.

These two examples are reasonably simple, but for linear spaces of higher dimensions it is by no means obvious how to generate an orthonormal basis. One way of formally generating an orthonormal basis from an arbitrary basis is to use the Gramm–Schmidt orthonormalisation process, and this is given later in this appendix

There are some further points that need to be aired before we get to discussing Fourier series proper. These concern the properties of bases, especially regarding linear spaces of infinite dimension. If the basis $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$ spans the linear space V, then *any* vector $\mathbf{v} \in V$ can be expressed as a linear combination of the basis vectors in the form

$$\mathbf{v} = \sum_{r=1}^{n} \alpha_r \mathbf{a}_r.$$

This result follows from the linear independence of the basis vectors, and that they span V.

If the basis is orthonormal, then a typical coefficient, α_k can be determined by taking the inner product of the vector $\mathbf{v}$ with the corresponding basis vector $\mathbf{a}_k$ as follows

$$\begin{aligned}\langle \mathbf{v}, \mathbf{a}_k \rangle &= \sum_{r=1}^{n} \alpha_r \langle \mathbf{a}_r, \mathbf{a}_k \rangle \\ &= \sum_{r=1}^{n} \alpha_r \delta_{kr} \\ &= a_k\end{aligned}$$

where δ_{kr} is the Kronecker delta:-

$$\delta_{kr} = \begin{cases} 1 & r = k \\ 0 & r \neq k \end{cases}.$$

If we try to generalise this to the case $n = \infty$ there are some difficulties. They are not insurmountable, but neither are they trivial. One extra need always arises when the case $n = \infty$ is considered and that is convergence. It is this that prevents the generalisation from being straightforward. The notion of *completeness* is also important. It has the following definition:

Definition C.9 *Let* $\{\mathbf{e}_1, \mathbf{e}_2, \ldots\ldots\}$ *be an infinite orthonormal system in an inner product space* V. *The system is* complete *in* V *if only the zero vector*$(\mathbf{u} = \mathbf{0})$ *satisfies the equation*

$$\langle \mathbf{u}, \mathbf{e}_n \rangle = \mathbf{0}, \quad n \in \mathbb{N}$$

A complete inner product space whose basis has infinitely many elements is called a Hilbert Space, the properties of which take us beyond the scope of this short appendix on linear algebra. The next step would be to move on to Bessel's Inequality which is stated but not proved in Chap. 4.

Here are a few more definitions that help when we have to deal with series of vectors rather than series of scalars

Definition C.10 *Let* $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n, \ldots$ *be an infinite sequence of vectors in a normed linear space (e.g. an inner product space)* W*. We say that the sequence* converges in norm *to the vector* $\mathbf{w} \in W$ *if*

$$\lim_{n \to \infty} ||\mathbf{w} - \mathbf{w}_n|| = \mathbf{0}.$$

This means that for each $\epsilon > 0$*, there exists* $n > n(\epsilon)$ *such that* $||\mathbf{w} - \mathbf{w}_n|| < \epsilon, \forall n > n(\epsilon)$.

Definition C.11 *Let* $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n, \ldots$ *be an infinite sequence of vectors in the normed linear space* V*. We say that the series*

$$\mathbf{w}_n = \sum_{r=1}^{n} \alpha_r \mathbf{a}_r$$

converges in norm *to the vector* $\mathbf{w}$ *if* $||\mathbf{w} - \mathbf{w}_n|| \to 0$ *as* $n \to \infty$*. We then write*

$$\mathbf{w} = \sum_{r=1}^{\infty} \alpha_r \mathbf{a}_r.$$

There is logic in this definition as $||\mathbf{w}_n - \mathbf{w}||$ measures the distance between the vectors $\mathbf{w}$ and $\mathbf{w}_n$ and if this gets smaller there is a sense in which $\mathbf{w}$ converges to $\mathbf{w}_n$.

Definition C.12 *If* $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n \ldots\}$ *is an infinite sequence of orthonormal vectors in a linear space* V *we say that the system is* closed in V *if, for every* $\mathbf{a} \in V$ *we have*

$$\lim_{n \to \infty} ||\mathbf{a} - \sum_{r=1}^{n} \langle \mathbf{a}, \mathbf{e}_r \rangle \mathbf{e}_r|| = 0.$$

There are many propositions that follow from these definitions, but for now we give one more definition that is useful in the context of Fourier series.

Definition C.13 *If* $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n \ldots\}$ *is an infinite sequence of orthonormal vectors in a linear space* V *of infinite dimension with an inner product, then we say that the system is* complete *in* V *if only the zero vector* $\mathbf{a} = \mathbf{0}$ *satisfies the equation*

$$\langle \mathbf{a}, \mathbf{e}_n \rangle = 0, \quad n \in \mathbb{N}.$$

There are many more general results and theorems on linear spaces that are useful to call on from within the study of Fourier series. However, in a book such as this a judgement has to be made as when to stop the theory. Enough theory of linear spaces has now been covered to enable Fourier series to be put in proper context. The reader who wishes to know more about the pure mathematics of particular spaces can enable their thirst to be quenched by many excellent texts on special spaces. (Banach Spaces and Sobolev Spaces both have a special place in applied mathematics, so inputting these names in the appropriate search engine should get results.)

C.2 Gramm–Schmidt Orthonormalisation Process

Even in an applied text such as this, it is important that we know formally how to construct an orthonormal set of basis vectors from a given basis. The Gramm–Schmidt process gives an infallible method of doing this. We state this process in the form of a theorem and prove it.

Theorem C.2 *Every finite dimensional inner product space has a basis consisting of orthonormal vectors.*

Proof Let $\{v_1, v_2, v_3, \ldots, v_n\}$ be a basis for the inner product space V. A second equally valid basis can be constructed from this basis as follows

$$\begin{aligned}
u_1 &= v_1 \\
u_2 &= v_2 - \frac{(v_2, u_1)}{||u_1||^2} u_1 \\
u_3 &= v_3 - \frac{(v_3, u_2)}{||u_2||^2} u_2 - \frac{(v_3, u_1)}{||u_1||^2} u_1 \\
&\vdots \\
u_n &= v_n - \frac{(v_n, u_{n-1})}{||u_{n-1}||^2} u_{n-1} - \cdots - \frac{(v_n, u_1)}{||u_1||^2} u_1
\end{aligned}$$

where $u_k \neq 0$ for all $k = 1, 2, \ldots, n$. If this has not been seen before, it may seem a cumbersome and rather odd construction; however, for every member of the new set $\{u_1, u_2, u_3, \ldots, u_n\}$ the terms consist of the corresponding member of the start basis $\{v_1, v_2, v_3, \ldots, v_n\}$ from which has been subtracted a series of terms. The coefficient of u_j in u_i $j < i$ is the inner product of v_i with respect to u_j divided by the length of u_j. The proof that the set $\{u_1, u_2, u_3, \ldots, u_n\}$ is orthogonal follows the standard induction method. It is so straightforward that it is left for the reader to complete. We now need two further steps. First, we show that $\{u_1, u_2, u_3, \ldots, u_n\}$ is a linearly independent set. Consider the linear combination

$$\alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n = 0$$

and take the inner product of this with the vector u_k to give the equation

$$\sum_{j=1}^{n} \alpha_j (u_j, u_k) = 0$$

from which we must have

$$\alpha_k (u_k, u_k) = 0$$

so that $\alpha_k = 0$ for all k. This establishes linear independence. Now the set $\{w_1, w_2, w_3, \ldots, w_n\}$ where

$$w_k = \frac{u_k}{||u_k||}$$

is at the same time, linearly independent, orthogonal and each element is of unit length. It is therefore the required orthonormal set of basis vectors for the inner product space. The proof is therefore complete. $\square$

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